5.6. Ratio Test and Root Test

Ratio Test

$$\sum_{n=1}^{\infty} a_n \quad \text{is a series.}$$

Consider the following limit $\rho = \lim_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

1. $0 \leq \rho < 1$, then the series converges absolutely.
2. $\rho > 1$, then the series diverges
3. $\rho = 1$, the test fails, i.e., it does not provide any information about the series. We have to use something else.

E.g. Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

Q: Test this series for absolute convergence?

Apply Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\boxed{a_n = (-1)^n \frac{n^3}{3^n}} \quad ; \quad a_{n+1} = (-1)^{n+1} \cdot \frac{(n+1)^3}{3^{n+1}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| (-1)^{n+1} \frac{(n+1)^3}{3^{\cancel{n+}1}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n}\, n^3} \right| = \left| \frac{(-1) \cdot (n+1)^3}{3 \cdot n^3} \right|$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{3n^3} . \quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \boxed{\frac{1}{3}} < 1 .$$

Conclusion: Series converges absolutely by the Ratio Test.

E.g. Consider $$\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}.$$

Test for absolute convergence.

Remind: $n! = 1\cdot 2\cdot 3\cdot 4\cdots n$; $(2n)! = 1\cdot 2\cdot 3\cdots (2n)$

Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right|$. $\quad a_n = \frac{(-1)^n (n!)^2}{(2n)!}$

$$a_{n+1} = \frac{(-1)^{n+1}[(n+1)!]^2}{(2(n+1))!} = \frac{(-1)^{n+1}[(n+1)!]^2}{(2n+2)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \frac{(n+1)\cancel{!}\,(n+1)\cancel{!}\,\overbrace{\cancel{(1\cdot 2\cdot 3\cdots(2n))}}^{(2n)!}}{\underbrace{\cancel{1\cdot 2\cdot 3\cdots(2n)}(2n+1)(2n+2)}_{(2n+2)!}\ \underbrace{\cancel{n!}\cdot\cancel{n!}}_{(n!)^2}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \boxed{\frac{1}{4}} < 1$$

Conclusion: series converges absolutely.

Ex: $$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

Test for absolute convergence (or divergence)

Apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)\cancel{!}}\cdot\frac{\cancel{n!}}{n^n} = \frac{(n+1)^{n+1}}{(n+1)\cdot n^n} = \frac{(n+1)^n\cdot\cancel{(n+1)}}{\cancel{(n+1)}\cdot n^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^n}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

From Cal 1, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$

Conclusion: Series diverges.

Reminder: $L = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Indeterminate form 1^∞

form: $\infty \cdot 0$

$$\ln L = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^{n} = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \overset{\frac{0}{0}}{\underset{\text{L'Hopital}}{=}} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

$\ln L = 1$. So, $\boxed{L = e}$

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$.

Consider the limit $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

(1) If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(2) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

(3) If $\rho = 1$, the test fails, i.e., it does not provide any information about the series.

E.g. Test convergence / divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

Apply the root test:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \boxed{\frac{2}{3}} < 1$$

Conclusion: the series converges absolutely.

E.g. $$\sum_{n=2}^{\infty} \frac{n^n}{(\ln(n))^n}$$

Apply the root test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^n}{(\ln(n))^n}} = \frac{n}{\ln(n)}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n}{\ln(n)} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$$

0

Conclusion: Series diverges.