6.1. Power Series

What is a power series?

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where c_n are constants.

E.g. If $c_0 = c_1 = c_2 = c_3 = \cdots = c_n = \cdots = 1$ ($c_n = 1$ for all n)

$$\underbrace{1 + x}_{x} \underbrace{+ x^2}_{x} \underbrace{+ x^3}_{x} + \cdots = \sum_{n=0}^{\infty} x^n$$

is an example of a power series

→ this is a geometric series with common ratio $= x$.

It converges if $|x| < 1$.

E.g. Consider the power series

$$\sum_{n=0}^{\infty} n!\, x^n \qquad (c_n = n!)$$

Q: For what values of x is the series convergent?

Apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\cancel{!}\ \cancel{x}^{n+1}}{\cancel{n!}\ \cancel{x^n}}\right|$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{1}\right| = \lim_{n\to\infty}|(n+1)x| = \infty > 1$$

The series diverges regardless of the values of x.

Except: when $x = 0$: the series $= 0$.

<u>E.g.</u> Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot (x-3)^n$$

$$\left(c_n = \frac{1}{n} \right)$$

This is an example of a power series centered at 3.

<u>Note:</u> $\sum_{n=0}^{\infty} c_n x^n$ is a power series center at 0

$\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at a

(Here a is constant)

<u>Back to E.g.</u> For what values of x does the series converge.

Apply Ratio Test: $a_n = \frac{(x-3)^n}{n}$; $a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(x-3)^n}}\right| = \left|\frac{n(x-3)}{n+1}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n(x-3)}{n+1}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot|x-3|$$

$$= |x-3| \longrightarrow \text{this is the limit}$$

By ratio test, series converges if $0 \leq \text{limit} < 1$.

Hence, series converges if $|x-3| < 1$.

Solve $|x-3| < 1$ for x.

$$|x-3| < 1 \iff -1 < x-3 < 1$$

$$2 < x < 4$$

The series will converge if $2 < x < 4$.

Note: the ratio test is inconclusive if limit $= 1$; i.e., if $x = 2$ or $x = 4$.

Hence, we do not know at this point whether the series converges or diverges at the endpoints $x = 2$ and $x = 4$. We need to plug $x = 2$ and $x = 4$ to the series and test for convergence / divergence separately.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}.$$

Plug $\boxed{x = 4}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This is a p-series $p = 1$.

(harmonic series)

It diverges.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}.$$

Plug $x = 2$ into the series.

We get: $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating Series. It converges by the alternating series Test.

Conclusion: Series converges when $2 < x < 4$ and $x = 2$

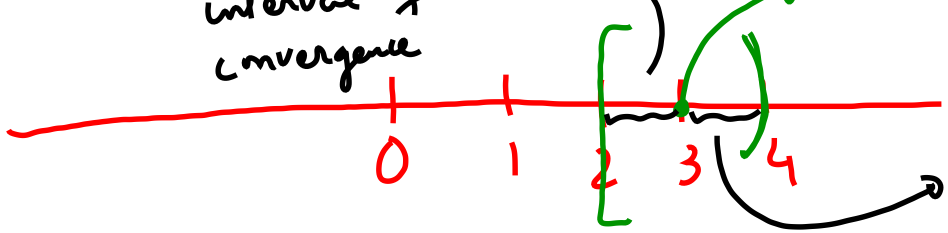
Series diverges for all other values of x.

x in the interval $[2, 4) \Longrightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges

The interval $[2, 4)$ is called the interval of convergence for the series.

interval of convergence

center of the series



distance from center to endpoints $= 1$

1 is called the Radius of Convergence

Theorem 6.1

Given any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$

Exactly one of the following scenarios will happen.

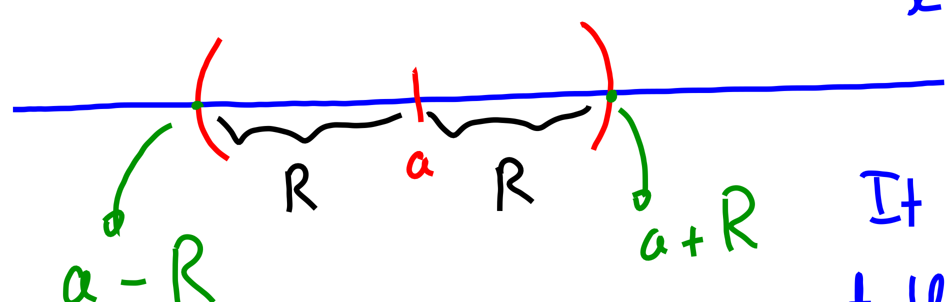
(1) The series converges only at the center $x = a$. the series diverges for all $x \neq a$.

(2) The series converges for all real numbers x.

(3) It converges on an interval surrounding the center a.

The series converges for every x in the interval $(a-R, a+R)$.

R is called the radius of convergence of the series



It may converge or diverge at the endpoints $a-R$; $a+R$.

E.g. Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Find the radius of convergence and interval of convergence for this series.

Apply ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right|$

$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot |x| = 0 < 1$

0

limit $= 0 < 1$ regardless of the values of x.

Therefore, the series converges for all values of x.

Interval of convergence: $(-\infty, \infty)$. Radius of convergence $= \infty$.

E.g. Find the radius of convergence and interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Apply Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} \cancel{x^{n+1}}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n} \cancel{x^n}}\right| = \left|\frac{-3\sqrt{n+1} \cdot x}{\sqrt{n+2}}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3\sqrt{n+1}}{\sqrt{n+2}} |x|.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n+2}} |x| = 3|x|$$

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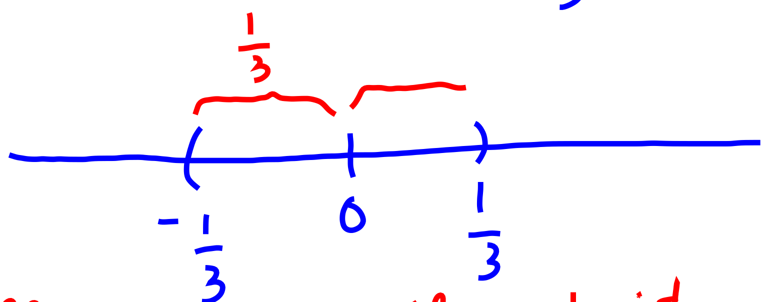
For series to converge, limit < 1.

Hence, $3|x| < 1$.

Solve this inequality $|x| < \frac{1}{3}$

$$-\frac{1}{3} < x < \frac{1}{3}.$$



Radius of Convergence $= \frac{1}{3}$.

* We still need to test the endpoints.

Plug $\boxed{x = \frac{1}{3}}$ into the series:
$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

This is an alternating series. By the A.S.T., series converges.

Plug $\boxed{x = -\frac{1}{3}}$ into the series.

$$\sum_{n=0}^{\infty} \frac{(-3)^n \cdot \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

Limit Comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 > 0$$

So, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ behaves exactly like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

p-series with $p = \frac{1}{2}$.

So it diverges.

Hence, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

Interval of convergence: $\boxed{\left[-\frac{1}{3}, \frac{1}{3}\right)}$

Represent Functions with Power Series.

<u>E.g.</u> $1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$

Geometric. Converges when $|x| < 1$

<u>Recall:</u> Sum of a geometric series $= \dfrac{\text{first term}}{1 - \text{common ratio}}$

Hence, $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x| < 1$

The function $f(x) = \dfrac{1}{1-x}$ is represented by the series

$$\sum_{n=0}^{\infty} x^n \text{ when } |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \; ; \; |x| < 1$$

$$\frac{1}{1-(\text{Stuff})} = 1 + (\text{Stuff}) + (\text{Stuff})^2 + \dots = \sum_{n=0}^{\infty} (\text{Stuff})^n \; ;$$

$$|\text{Stuff}| < 1$$

E.g. Use a power series to represent the given function. Find the interval of convergence.

① $f(x) = \dfrac{1}{1+x^3}$ ② $g(x) = \dfrac{x^2}{4-x^2}$.

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① $\dfrac{1}{1-(-x^3)} = 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \cdots$

Stuff

$= \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{3n}$

It converges if $|-x^3| < 1$; $|x|^3 < 1$; $-1 < x < 1$

② $\dfrac{x^2}{4-x^2} = \dfrac{x^2}{4\left(1-\frac{x^2}{4}\right)} = \dfrac{x^2}{4}\cdot \dfrac{1}{1-\frac{x^2}{4}}$

series converges if

$\left|\dfrac{x^2}{4}\right| < 1$

$|x^2| < 4$

$|x| < 2$; $-2 < x < 2$

$$= \frac{x^2}{4}\cdot\left(1 + \left(\frac{x^2}{4}\right) + \left(\frac{x^2}{4}\right)^2 + \left(\frac{x^2}{4}\right)^3 + \cdots\right)$$

$$= \frac{x^2}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n = \frac{x^2}{4}\cdot\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n}$$

$$= \sum_{n=0}^{\infty}\frac{x^2}{4}\cdot\frac{x^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{x^{2n+2}}{4^{n+1}}$$

this is the power series which represent the function g.

Interval of convergence $(-2,2)$