6.2. Properties of Power Series

Goals: (1) Add, Subtract, Multiply Power Series

(2) Differentiate and Integrate Power Series

Power Series centered at a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Power Series for $f(x) = \frac{1}{1-x}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \; ; \; |x| < 1$$

$$\frac{1}{1-(\text{Stuff})} = \sum_{n=0}^{\infty} (\text{Stuff})^n \; ; \; |\text{Stuff}| < 1$$

E.g. $f(x) = \dfrac{3x}{1+x^2}$.

Find a power series for f. Find the interval of convergence for the series.

$$f(x) = 3x \cdot \frac{1}{1+x^2} = 3x \cdot \left(\frac{1}{\underbrace{1-(-x^2)}} \right)$$

$$= 3x \cdot \sum_{n=0}^{\infty} (-x^2)^n = 3x \cdot \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}}$$

converges if $|-x^2| < 1$

$|x|^2 < 1$; $|x| < 1$

x is in $(-1, 1)$

Add / Subtract Power Series

Suppose $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to functions f and g on the same interval I.

Then $\underbrace{\sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} d_n x^n}_{\sum_{n=0}^{\infty} (c_n + d_n) x^n}$ converges to $f+g$ on I.

<u>E.g.</u> HW # 1. $f(x) = \dfrac{130}{(x^2+1)(x^2-25)}$.

Find power series for $f(x)$

Partial Fraction Decomposition for

$$f(x) = \frac{130}{(x^2+1)(x^2-25)} = \frac{130}{(x^2+1)(x-5)(x+5)}$$

$$= \frac{Ax+B}{x^2+1} + \frac{C}{x-5} + \frac{D}{x+5}$$

$A = 0$; $B = -5$ $C = +\frac{1}{2}$ $D = -\frac{1}{2}$

$$f(x) = \frac{-5}{1+x^2} + \frac{1/2}{-5+x} + \frac{-1/2}{5+x}$$

$$\frac{-5}{1+x^2} = \frac{-5}{1-(-x^2)} = -5 \cdot \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} -5 \cdot (-1)^n \cdot x^{2n}$$

converges when $|-x^2| < 1$; $|x|^2 < 1$; $|x| < 1$

$$\frac{1/2}{-5+x} = \frac{1/2}{-5\left(1-\frac{x}{5}\right)} = -\frac{1}{10}\cdot\boxed{\frac{1}{1-\frac{x}{5}}}$$

$$= -\frac{1}{10}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{10}\right)\cdot\frac{x^n}{5^n}$$

converges when $\left|\frac{x}{5}\right| < 1$; $|x| < 5$

$$\frac{-1/2}{5+x} = \frac{-1/2}{5\left(1+\frac{x}{5}\right)} = \frac{-1/2}{5\left(1-\left(-\frac{x}{5}\right)\right)} = -\frac{1}{10}\cdot\underline{\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n}$$

$$= \sum_{n=0}^{\infty}\overbrace{\boxed{\left(-\frac{1}{10}\right)\cdot\frac{(-1)^n}{5^n}}}^{c_n}x^n$$; converges when $\left|-\frac{x}{5}\right| < 1$; $|x| < 5$

Common interval of convergence of these 3 series is $(-1, 1)$

$$f(x) = \sum_{n=0}^{\infty}(-5)\cdot(-1)^n\cdot x^{2n} + \underbrace{\sum_{n=0}^{\infty}\left(-\frac{1}{10}\right)\frac{x^n}{5^n} + \sum_{n=0}^{\infty}\left(-\frac{1}{10}\right)\frac{(-1)^n}{5^n}x^n}_{\text{converges when } |x| < 1}$$

$$f(x) = \sum_{n=0}^{\infty} (-5)(-1)^n x^{2n} + \sum_{n=0}^{\infty} \left(\left(-\frac{1}{10}\right) \frac{1}{5^n} + \left(-\frac{1}{10}\right) \cdot \frac{(-1)^n}{5^n} \right) x^n$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{10} \cdot \frac{1}{5^n} \left(1 + (-1)^n \right) \right) x^n$$

$$= \sum_{n=0}^{\infty} \left((-5)(-1)^n x^{2n} + \left(-\frac{1}{10} \cdot \frac{1}{5^n} \left(1 + (-1)^n \right) \right) x^n \right)$$

Note: In WebAssign, decompose function as:

$$\frac{130}{(x^2+1)(x^2-25)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-25}.$$

Multiply Power Series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots$$

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right)$$

$$= \left(\boxed{c_0} + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right)\left(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots\right)$$

$$= c_0 d_0 + \left(c_0 d_1 + c_1 d_0\right) x + \left(c_0 d_2 + c_1 d_1 + c_2 d_0\right) x^2$$

$$\left(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0\right) x^3 + \cdots$$

In general,

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0$$

Term-by-term differentiation and Integration of Power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{converges} \quad |x-a| < R$$

$a-R \quad a \quad a+R$

Then

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x-a)^n\right)$$

$$= \sum_{n=0}^{\infty} c_n \frac{d}{dx}\left[(x-a)^n\right] = \boxed{\sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$ converges for $|x-a| < R$

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$$

$$= \sum_{n=0}^{\infty} c_n \underbrace{\int (x-a)^n dx}$$

$u = x-a$

$du = dx$

$$\int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x-a)^{n+1}}{n+1} + C$$

$$= \left(\sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1} \right) + C$$

E.g. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \,; \quad |x| < 1$

Find power series for $\dfrac{1}{(1-x)^2}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = -\frac{-1}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left(x^n\right) = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}} \,; \text{ converges when } |x| < 1$$

$\underline{\text{E.g.}}$ $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n .$$

$$\underbrace{\int \frac{1}{1+x}\,dx} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \underbrace{\int x^n \, dx}$$

$$\underline{\ln(1+x)} = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C} \; ; \quad |x| < 1$$

x is $(-1, 1)$

What is the constant C ?

Plug $x = 0$ into both sides

$$\ln 1 = C \; ; \quad \text{So, } C = 0$$

$$\ln(1+x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}} \; ; \quad |x| < 1$$

E.g. $$\frac{1}{1+x^2} = \frac{1}{1-\boxed{(-x^2)}} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

converges $|x| < 1$.

$$\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$\arctan x = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} + C \quad , \; |x| < 1$$

Plug in $x = 0$; $C = 0$

HW #2

Find power series for $f(x) = \dfrac{x^7}{(8+x^8)^2}$

using differentiation.

$$g(x) = \frac{1}{8+x^8} = \frac{1}{8\left(1+\frac{x^8}{8}\right)} = \frac{1}{8\left(1-\left(-\frac{x^8}{8}\right)\right)}$$

$$\frac{1}{8+x^8} = \frac{1}{8}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^8}{8}\right)^n \qquad \left(\text{converges } \left|\frac{x^8}{8}\right| < 1 \quad |x|^8 < 8 \quad |x| < \sqrt[8]{8}\right)$$

$$\frac{1}{8+x^8} = \sum_{n=0}^{\infty}\frac{1}{8}\cdot\frac{(-1)^n}{8^n}\cdot x^{8n}$$

Differentiate both sides:

$$\frac{d}{dx}\left(\frac{1}{8+x^8}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{8} \cdot \frac{(-1)^n}{8^n} \cdot x^{8n}\right)$$

$$-\frac{8x^7}{(8+x^8)^2} = \sum_{n=0}^{\infty} \frac{1}{8} \cdot \frac{(-1)^n}{8^n} \cdot \frac{d}{dx}\left(x^{8n}\right)$$

Multiply by $-\frac{1}{8}$ to both sides

$$\frac{x^7}{(8+x^8)^7} = -\frac{1}{8} \cdot \sum_{n=1}^{\infty} \frac{1}{\cancel{8}} \frac{(-1)^n}{8^n} \cdot \cancel{8} n x^{8n-1}$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right) \cdot \frac{(-1)^n}{8^n} n x^{8n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{8^{n+1}} n x^{8n-1} = \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^{n+1} \cdot n x^{8n-1}$$

#7. Find $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = ?$

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad ; \; |x| < 1$$

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad ; \; |x| < 1$$

$$f'\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\frac{1}{4}} = 4$$

Multiply by $\frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$

#8. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$

Integrate both sides from 0 to 1

$$\sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n}\,dx = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}\Bigg|_0^1 = \arctan(x)\Big|_0^1$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{2n+1}\right) = \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4}$$