6.3. Taylor Series and Maclaurin Series

We have seen

$$\frac{1}{1-(\text{Stuff})} = \sum_{n=0}^{\infty} (\text{Stuff})^n \quad ; \quad |\text{Stuff}| < 1$$

We can also differentiate and integrate this term-by-term to obtain power series for many functions.

Taylor Series

Theorem: If f has a power series expansion centered at a, that is,

if $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad ; \quad |x-a| < R$$

Then the coefficients c_n are given by the formula:

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

In other words, the Taylor series for $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

In the special case that the center $a = 0$; the Taylor series for f has the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

This is called the Maclaurin series for f.

(Taylor series when center $a=0$)

E.g. Find the Maclaurin series and its radius of convergence for the given function.

ⓐ $f(x) = e^x$ ⓑ $g(x) = \sin x$ ⓒ $h(x) = \cos x$

ⓐ $f(x) = e^x$. Find Maclaurin series & its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$f(0) = 1$; $f'(x) = e^x$; $f'(0) = 1$; $f''(x) = e^x$; $f''(0) = 1$

In general, $f^{(n)}(x) = e^x$; $f^{(n)}(0) = 1$; for all $n \geq 1$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Radius of Convergence of this series?

Ratio Test: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right|$

$$= \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1$$

Conclusion: Series converges regardless of what x is.

In other words, it converges for all values of x.

So, Radius of Convergence $= \infty$.

So, $$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x}$$

An implication of this: $e^{1.5} = 4.48169\ldots$

$$\sum_{n=0}^{1000} \frac{(1.5)^n}{n!} \simeq 4.481689$$

$$\sum_{n=0}^{10000} \frac{(1.5)^n}{n!} \simeq 4.48169$$

$$\sum_{n=0}^{1} \frac{x^n}{n!} = 1 + x \ ; \ \sum_{n=0}^{2} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} \ ; \ \sum_{n=0}^{3} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

(b) $g(x) = \sin x$

$g(0) = 0$

$g'(x) = \cos x \; ; \; g'(0) = 1$

$g''(x) = -\sin x \; ; \; g''(0) = 0$

$g'''(x) = -\cos x \; ; \; g'''(0) = -1$

$g^{(4)}(x) = \sin x \; ; \; g^{(4)}(0) = 0$

$g^{(5)}(0) = 1$

$g^{(6)}(0) = 0$

$g^{(7)}(0) = -1$

$g^{(8)}(0) = 0$

$g^{(9)}(0) = 1$

$g^{(10)}(0) = 0$

$g^{(11)}(0) = -1$

$g^{(12)}(0) = 0$

$g^{(2k)}(0) = 0 \; ; \quad g^{(2k+1)}(0) = (-1)^k$

$$\sin x = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\boxed{\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$ for all x

Radius of Convergence : ∞ (by ratio test)

ⓒ $h(x) = \cos x$

$h^{(2k)}(0) = (-1)^k$; $h^{(2k+1)}(0) = 0$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\boxed{\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{for all } x \quad (\text{by Ratio Test})}$$

Function	MacLaurin Series	Convergence
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	all x
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	all x
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	all x
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x \rvert < 1$ (or $-1 < x < 1$)
$f(x) = \arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	$\lvert x \rvert < 1$ (or $-1 < x < 1$)

Taylor Polynomials

Suppose $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The n^{th} degree Taylor polynomial for f is the n^{th} partial sum of the Taylor series for f.

0^{th} poly: $P_0(x) = f(a)$

1^{st} poly: $P_1(x) = f(a) + f'(a)(x-a)$

2^{nd} poly: $P_2(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)(x-a)^2}{2!}$

In general,

The n^{th} degree Taylor polynomial is

$$\boxed{p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

2 results regarding Taylor polynomials.

Taylor Remainder Theorem.

f : differentiable $(n+1)$ times on an interval I containing a.

$$\underbrace{R_n(x)}_{n^{\text{th}} \text{ remainder}} = f(x) - p_n(x)$$

1 The Taylor series for f converges to f if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

② If there is a number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

E.g. $f(x) = \sqrt{x}$.

① Find the 1^{st} and 2^{nd} degree Taylor polynomials for f centered at $a = 4$.

② Use these polynomials to estimate $f(6) = \sqrt{6}$. Use Taylor's Theorem to bound the errors.

① Find $p_1(x)$; $p_2(x)$

$$p_1(x) = f(a) + f'(a)(x-a)$$

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$f(x) = \sqrt{x}$; $a = 4$

$f(4) = 2$; $f'(x) = \frac{1}{2\sqrt{x}}$; $f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4x^{3/2}}$; $f''(4) = -\frac{1}{4\cdot(4)^{3/2}} = -\frac{1}{32}$

$$p_1(x) = 2 + \frac{1}{4}(x-4)$$

$$p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

② Use $p_1(x)$; $p_2(x)$ to approximate $f(6)$.

$p_1(6) = 2 + \frac{1}{4}(6-4) = 2.5$; $p_2(6) = 2 + \frac{1}{4}(6-4) - \frac{1}{64}(6-4)^2$

$= 2.4375$

* Error bounds for these approximation.

$$|R_1(x)| \leq \frac{M}{2!}|x-a|^2 ;\ M \text{ is such that } |f''(x)| \leq M \text{ in } I$$

$$|R_2(x)| \leq \frac{M}{3!}|x-a|^3 ;\ M \text{ is such that } |f'''(x)| \leq M \text{ in } I.$$

* In this case, $x=6$; $a=4$; $I=(4,6)$

$$|f''(x)| = \left|-\frac{1}{4x^{3/2}}\right| = \left|\frac{1}{4x^{3/2}}\right|$$

What is the upper bound of $|f''(x)|$ on $(4,6)$? $|f''(4)| = \left|\frac{1}{4\cdot 4^{3/2}}\right|$

$$= \frac{1}{32}.$$

$$|R_1(6)| \leq \frac{\frac{1}{32}}{2}\cdot|6-4|^2 = 0.0625$$

This says that when we use $p_1(6)$ to estimate $f(6)$, the remainder $R_1(6)$ is at most 0.0625