6.4. Working with Taylor / Maclaurin Series

Maclaurin Series for $f(x) = (1+x)^r$; r : a real number.

Note: The Maclaurin series for $f(x) = (1+x)^r$ is called the binomial series.

Formula:

$$\boxed{(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \text{ ; converges for } |x| < 1}$$

Here $\binom{r}{n}$ is called the binomial coefficient:

$$\boxed{\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}}$$

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

E.g. $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

① Find the binomial series for $f(x)$

② Find the 3^{rd} degree Taylor polynomial $p_3(x)$ and use it to estimate $f(0.5) = \sqrt{1.5}$. And use Taylor's Remainder Theorem to bound the error.

① $$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots + \frac{r(r-1)(r-2)\dots(r-n+1)}{n!}x^n + \dots$$

Here $r = \frac{1}{2}$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \dots$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{2!\,2^2} + \frac{1\cdot 3}{3!\,2^3}x^3 - \frac{1\cdot 3\cdot 5}{4!\,2^4}x^4 + \dots + (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\,2^n}x^n + \dots$$

2

$$P_3(x) = 1 + \frac{x}{2} - \frac{x^2}{2!\,2^2} + \frac{1\cdot 3}{3!\,2^3}x^3$$

$$P_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

$$\underbrace{f(0.5)}_{\sqrt{1.5}} \approx P_3(0.5) = 1 + \frac{0.5}{2} - \frac{(0.5)^2}{8} + \frac{(0.5)^3}{16} \approx 1.2266$$

Error bound: $|R_3(x)| \leq \frac{M}{4!}|x-a|^4$; M: upper bound for $f^{(4)}(x)$ on I

$$f^{(4)}(x) = -\frac{15}{16(1+x)^{7/2}}.$$

$x = 0.5$; $a = 0$

$I = (0, 0.5)$; $|f^{(4)}(x)| \leq \boxed{\frac{15}{16}}$ M

$$\left|R_3(0.5)\right| \le \frac{15}{(16)4!}\cdot(0.5)^4 \approx 0.00244$$

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$-\infty < x < \infty$

$f(x) = (1+x)^{r}$	$\displaystyle\sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \dots + \frac{r(r-1)\dots(r-n+1)}{n!}x^n + \dots$	$-1 < x < 1$
$f(x) = \tan^{-1}(x)$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	$-1 < x \leq 1$
$f(x) = \ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x \leq 1$

Recall: $$\int \frac{1}{1+x}\,dx = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} \int (-x)^n\,dx \quad ; \quad -1 < x < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad ; \quad -1 < x \leq 1$$

Derive Maclaurin Series for functions using known series.

(a) Find the Maclaurin Series for $f(x) = \cos(\sqrt{x})$

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

converges on the domain of $\cos(\sqrt{x})$

which means $x \geq 0$

(b) Find the Maclaurin Series for

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$f(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \frac{1}{2} \sum_{n=0}^{\infty}\left(\frac{x^n}{n!} - \frac{(-1)^n x^n}{n!}\right)$$

If n is even, $(-1)^n = 1$. So, the terms corresponding to n even are all zero

When n is odd, $n = 2m+1$

$$\frac{1}{2}\left(\frac{x^{2m+1}}{(2m+1)!} - \frac{(-1)\cdot x^{2m+1}}{(2m+1)!}\right) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x^{2m+1}}{(2m+1)!} = \frac{x^{2m+1}}{(2m+1)!}$$

Series $\displaystyle\sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}$

Differentiate or Integrate known series to find series for other functions

E.g. $f(x) = \sqrt{1+x} = (1+x)^{1/2} = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{1.3.5\cdots(2n-3)}{2^n}x^n$

Differentiate to find the series for $\boxed{g(x) = \frac{1}{\sqrt{1+x}}}$

$$\boxed{f'(x) = \frac{1}{2\sqrt{1+x}}}$$

So $g(x) = 2f'(x)$

$$g(x) = 2 \cdot \frac{d}{dx}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^n\right)$$

$$= 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} \cdot n\, x^{n-1}$$

$$= 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{n-1}$$

Use power series to solve differential equations

E.g. Find a function $y = f(x)$ such that $y' = y$; $y(0) = 3$

Sol: Power series for y is $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

$y' = y$; $\underbrace{c_1 + 2c_2 x + 3c_3 x^2 + \cdots}_{y'} = \underbrace{c_0 + c_1 x + c_2 x^2 + \cdots}_{y}$

$c_0 = c_1$

$c_1 = 2c_2$

$c_2 = 3c_3$

$c_3 = 4c_4$

$\vdots$

$y(0) = 3$.

$y = c_0 + c_1 x + c_2 x^2 + \cdots$

$y(0) = c_0$

$\Rightarrow c_0 = 3$

$c_1 = 3$; $c_2 = \frac{3}{2}$; $c_3 = \frac{3}{2 \cdot 3}$; $c_4 = \frac{3}{2 \cdot 3 \cdot 4}$.

$c_n = \frac{3}{n!}$

e^x

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{3}{n!} x^n = 3 \sum_{n=0}^{\infty} \frac{x^n}{n!} = 3e^x$$