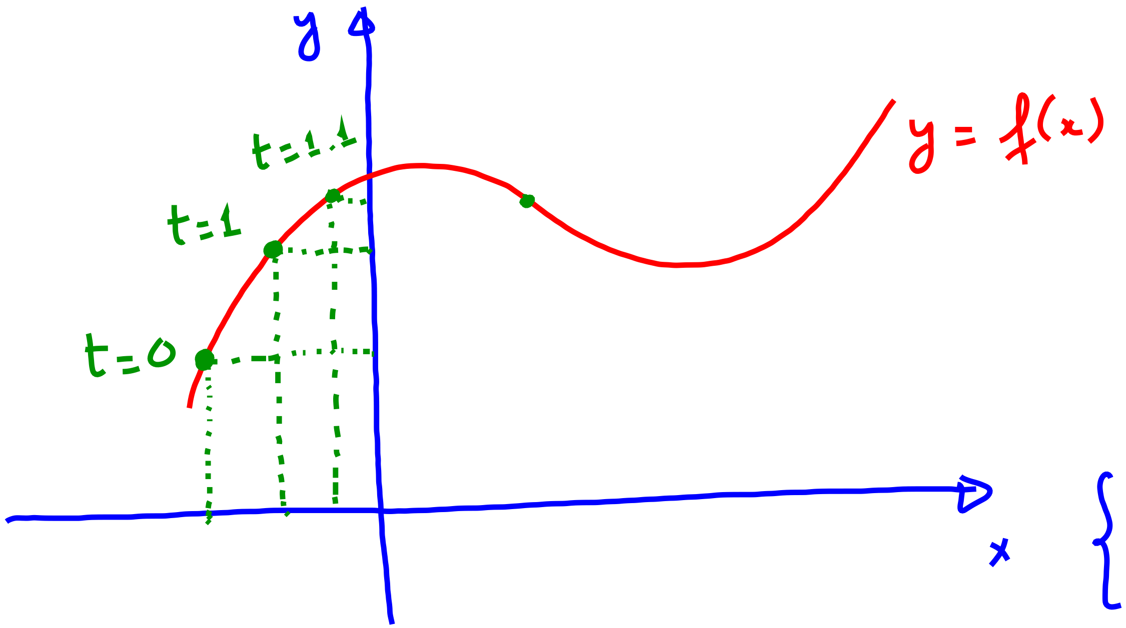
7.1 and 7.2 - Parametric Curves and Calculus of Parametric Curves.

What is a parametric curve?



x and y coordinates of a point moving on this curve change with respect to time. Hence they are functions of time.

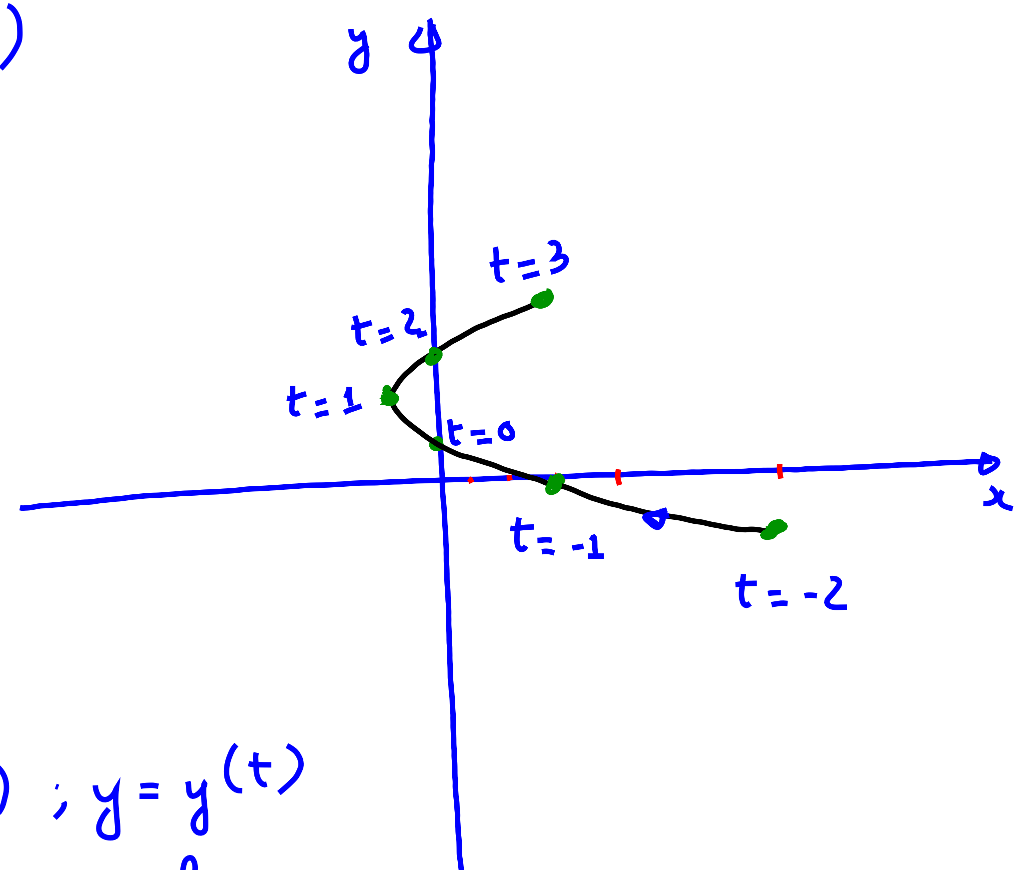
$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

|| these are called parametric equations

E.g. $x(t) = t^2 - 2t$; $y(t) = t + 1$

$\longrightarrow$ parametric described by 2 parametric equations

t	$x = t^2 - 2t$	$y = t+1$	
-2	8	-1	$\to (8,-1)$
-1	3	0	$\to (3,0)$
0	0	1	$\to (0,1)$
1	-1	2	$\to (-1,2)$
2	0	3	$\to (0,3)$
3	3	4	$\to (3,4)$



Eliminate the parameter. $x = x(t)$; $y = y(t)$

Solve for t in terms of x or y.

Plug that into the other equation.

$$x = t^2 - 2t \quad ; \quad y = t + 1$$

Solve for t in terms of y in the second equation: $t = y - 1$

Plug $t = y - 1$ into first equation: $x = (y-1)^2 - 2(y-1)$

$x = y^2 - 2y + 1 - 2y + 2$

$x = \underbrace{y^2 - 4y + 3}_{\text{parabola}}$.

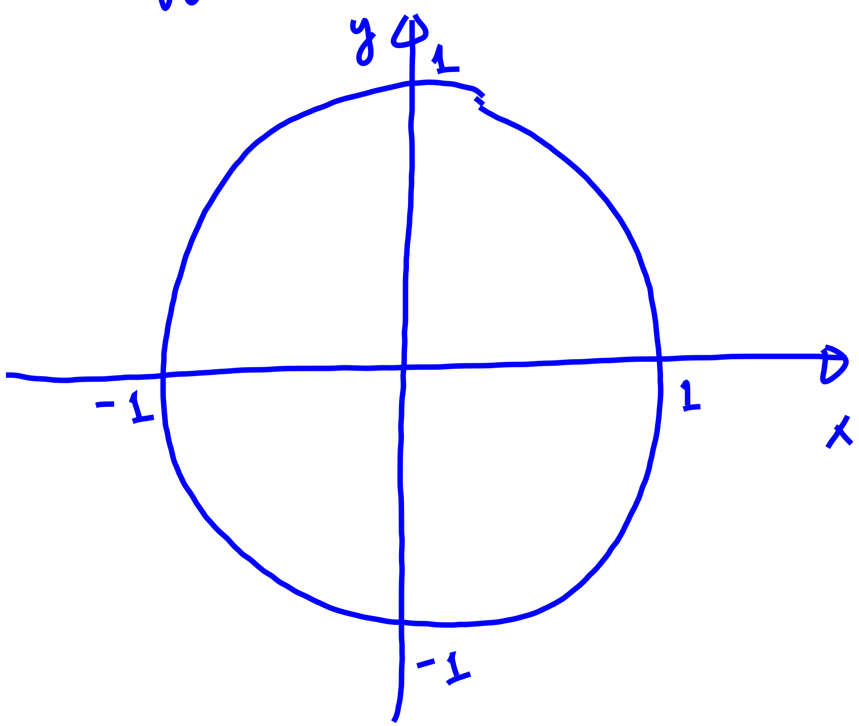
Vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$y = \frac{4}{2} = 2$; $x = -1$

$(-1, 2) \longrightarrow$ vertex

E.g. $x = \cos t$; $y = \sin t$; $0 \leq t \leq 2\pi$.

Which curve do these parametric equations represent?



$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

Calculus of Parametric Curves

Tangent Problem

Consider a curve defined by the parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

Suppose $x'(t)$, $y'(t)$ exist and $x'(t) \neq 0$.

Then $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Suppose $x'(t)$, $y'(t)$ exist and $y'(t) \neq 0$

$$\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{x'(t)}{y'(t)}$$

E.g. * Find the derivative $\frac{dy}{dx}$ for the curve defined by the equations

$$x = x(t) = t^2 - 4t \ ; \quad y = y(t) = 2t^3 - 6t \ ; \quad -2 \leq t \leq 3$$

* locate any critical points on the graph.

Sol: $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{6t^2 - 6}{2t - 4} = \frac{6(t^2 - 1)}{2(t-2)} = \frac{6(t-1)(t+1)}{2(t-2)}$$

* Recall: critical points are points where $\frac{dy}{dx}$ is zero or undefined.

$$\frac{dy}{dx} = 0 \text{ when } 6(t-1)(t+1) = 0.$$
$$t = 1 \ ; \ t = -1$$

When $t = 1$; $x = -3$; $y = -4$. $\longrightarrow (-3, -4)$

When $t = -1$; $x = 5$; $y = 4 \longrightarrow (5, 4)$

$\frac{dy}{dx}$ is undefined when $2(t-2) = 0 \longrightarrow t = 2$. When $t = 2$; $x = -4$; $y = 4$

$(-4, 4)$

Find the equation of a tangent line.

$$x = t^2 - 4t \;;\; y = 2t^3 - 6t \;;\quad -2 \le t \le 6$$

Find the equation of the tangent line to the curve at the point where $t = 5$.

<u>Recall</u>: Slope of tangent line at point $= \frac{dy}{dx}$ at that point.

$$\frac{dy}{dx} = \frac{6(t-1)(t+1)}{2(t-2)}$$

Slope

When $t = 5$; $\frac{dy}{dx} = \frac{6\cdot(5-1)(5+1)}{2\cdot(5-2)} = \frac{6\cdot 4\cdot 6}{2\cdot 3} = \boxed{24}$

Point: when $t = 5$; $x = (5)^2 - 4(5) = 5$ Point $(5, 220)$

$y = 2(5)^3 - 6(5) = 220$

Point - slope equation:

$$y - 220 = 24(x - 5)$$

$$y - 220 = 24x - 120$$

$$\boxed{y = 24x + 100}$$

Find the second derivative using parametric equations

$\dfrac{dy}{dx} \rightarrow$ 1st derivative

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) \rightarrow$ 2nd derivative

$$\underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}_{\frac{d^2y}{dx^2}} \qquad \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\boxed{\frac{d}{dt}\left(\boxed{\frac{dy}{dx}}\right)}}{\boxed{dx/dt}}$$

Process of finding 2nd derivative from parametric equations

1. $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$
2. Find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$
3. Divide by $x'(t)$

E.g. $x = t^2 - 3$; $y = 2t - 1$.

Find $\dfrac{d^2y}{dx^2} = ?$

1. $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{2}{2t} = \dfrac{1}{t}$.

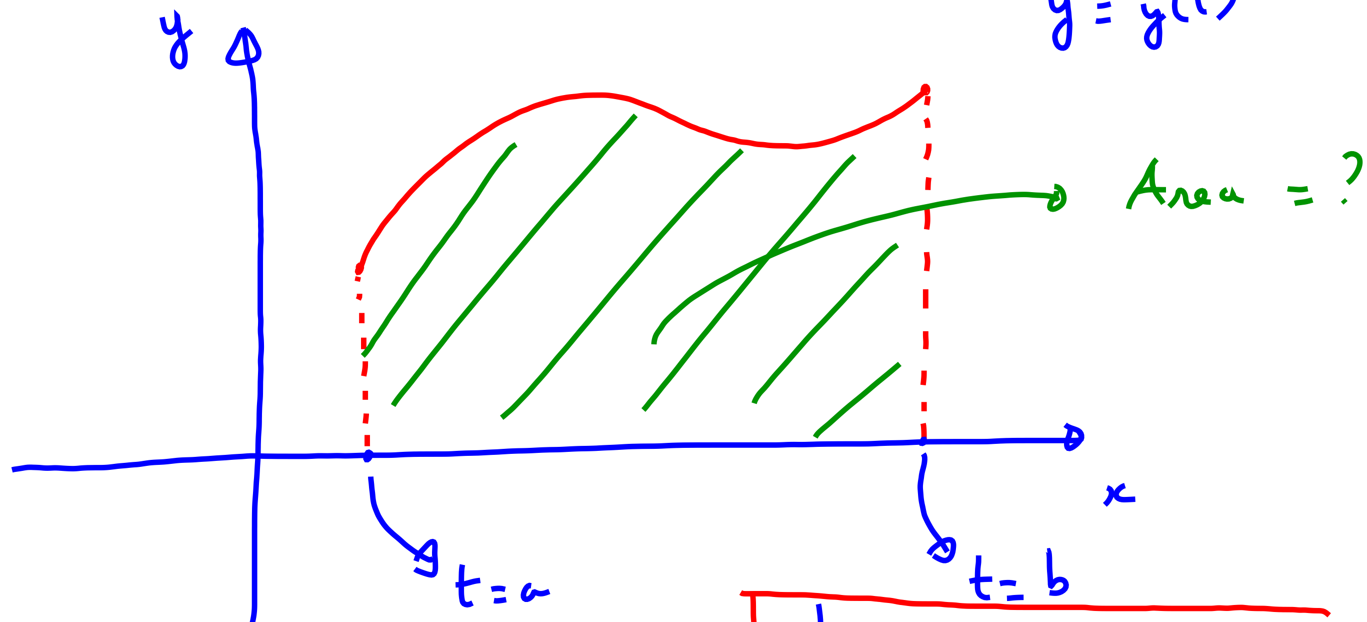
2. $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{1}{t}\right) = -\dfrac{1}{t^2}$.

3. $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \dfrac{-\dfrac{1}{t^2}}{2t} = \boxed{-\dfrac{1}{2t^3}}$

Area under a parametric curve

$x = x(t)$

$y = y(t)$

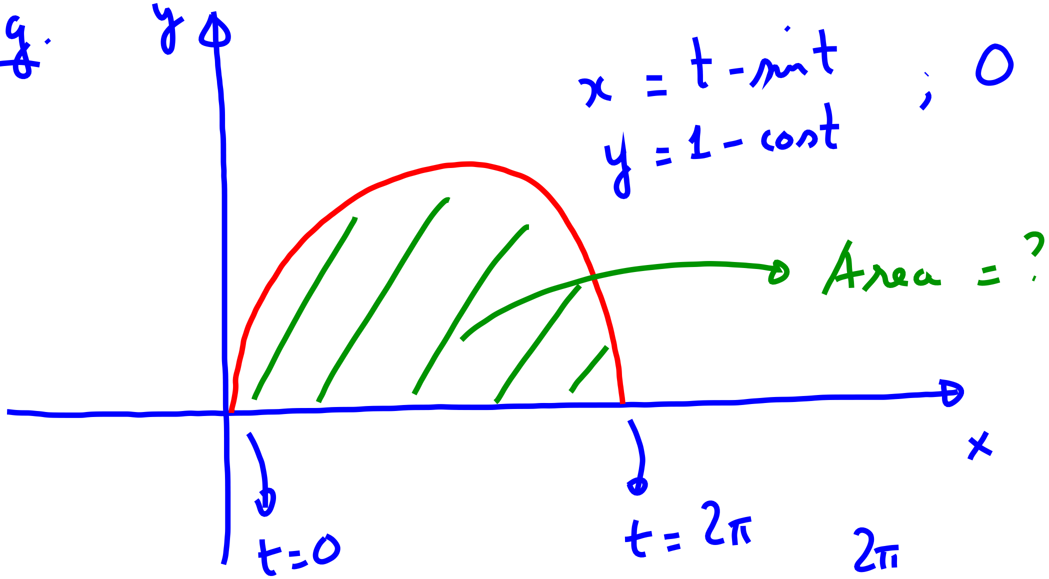


$$\text{Area} = \int y\, dx = \boxed{\int_a^b y(t) \cdot x'(t)\, dt}$$

$$dx = x'(t)\, dt$$

E.g.

$$x = t - \sin t, \quad y = 1 - \cos t \quad ; \quad 0 \le t \le 2\pi$$



$$\text{Area} = \int_a^b y(t)\, x'(t)\, dt = \int_0^{2\pi} (1-\cos t)(1-\cos t)\, dt$$

$$= \int_0^{2\pi} \left(1 - 2\cos t + \boxed{\cos^2 t}\right) dt = \int_0^{2\pi} \left(1 - 2\cos t + \boxed{\frac{1+\cos(2t)}{2}}\right) dt$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt = \left(\frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4}\right)\Bigg|_0^{2\pi}$$

$$= \boxed{3\pi}$$

Ex.

#12 HW $x = \cos(t)$; $y = e^t$; $0 \leq t \leq \frac{\pi}{2}$.

Find area under the curve defined above.

$$\text{Area} = \int_0^{\pi/2} e^t \cdot (-\sin t)\, dt = -\int_0^{\pi/2} e^t \sin t \, dt$$

Integration by Parts : L I A T E

$$\begin{cases} u = \sin t \\ dv = e^t \, dt \end{cases} \longrightarrow \begin{cases} du = \cos t \, dt \\ v = e^t \end{cases}$$

$$-\int_0^{\pi/2} e^t \sin t \, dt = -\left[e^t \sin t \Big|_0^{\pi/2} - \int_0^{\pi/2} e^t \cos t \, dt \right]$$

$$-\int_0^{\pi/2} e^t \sin t \, dt = -\left[e^{\pi/2} - \int_0^{\pi/2} e^t \cos t \, dt \right]$$

$$= -e^{\pi/2} + \underbrace{\int_0^{\pi/2} e^t \cos t \, dt}$$

$$\begin{cases} u = \cos t \\ dv = e^t \, dt \end{cases} \longrightarrow \begin{cases} du = -\sin t \, dt \\ v = e^t \end{cases}$$

$$-\int_0^{\pi/2} e^t \sin t \, dt = -e^{\pi/2} + e^t \cos t \Big|_0^{\pi/2} + \boxed{\int_0^{\pi/2} e^t \sin t \, dt}$$

$$-2\int_0^{\pi/2} e^t \sin t \, dt = -e^{\pi/2} - 1 \; ; \qquad \int_0^{\pi/2} e^t \sin t \, dt = \boxed{\frac{e^{\pi/2} + 1}{2}}$$

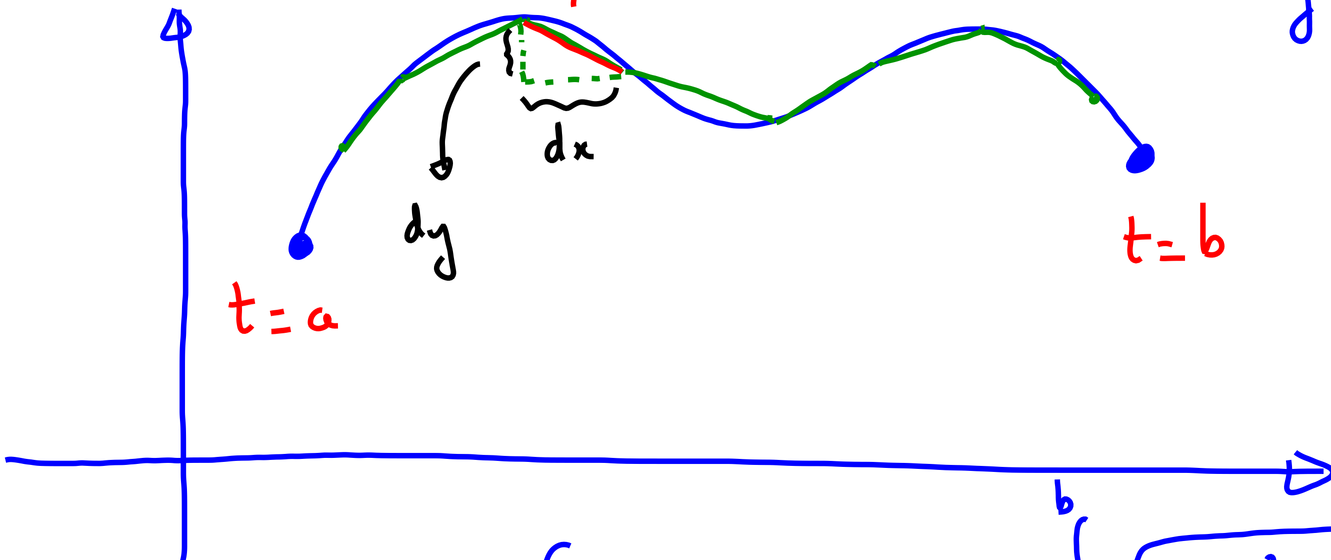
Arc length

$$\sqrt{(dx)^2+(dy)^2} = ds$$

$x = x(t)$

$y = y(t)$

$a \le t \le b$



$$\text{Arc length} = \int \sqrt{(dx)^2+(dy)^2} = \int_a^b \sqrt{\frac{(dx)^2+(dy)^2}{(dt)^2}\cdot(dt)^2}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

The length of the curve defined by the parametric equations

$$x = x(t) \; ; \; y = y(t) \; ; \; a \leq t \leq b$$

is given by the formula:

$$\text{Length} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt$$

E.g. Find the length of the curve defined by the parametric equations

$$x = 3\cos t \; ; \; y = 3\sin t, \; 0 \leq t \leq \pi.$$

Sol: $x'(t) = -3\sin t \; ; \; y'(t) = 3\cos t$

$$\text{length} = \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2} \, dt$$

$$= \int_0^\pi \sqrt{9\sin^2 t + 9\cos^2 t} \, dt = 3\int_0^\pi dt = \boxed{3\pi}$$

Ex. Find the arc length of the curve defined by

$$x = 3t^2 \;;\quad y = 2t^3 \;;\quad 1 \leq t \leq 3.$$

<u>Sol:</u> $x'(t) = 6t$; $y'(t) = 6t^2$

$$\text{length} = \int_1^3 \sqrt{(6t)^2 + (6t^2)^2}\; dt$$

$$= \int_1^3 \sqrt{36t^2 + 36t^4}\; dt = 6\int_1^3 \sqrt{t^2(1+t^2)}\; dt$$

$$= 6\int_1^3 t\sqrt{1+t^2}\; dt.$$

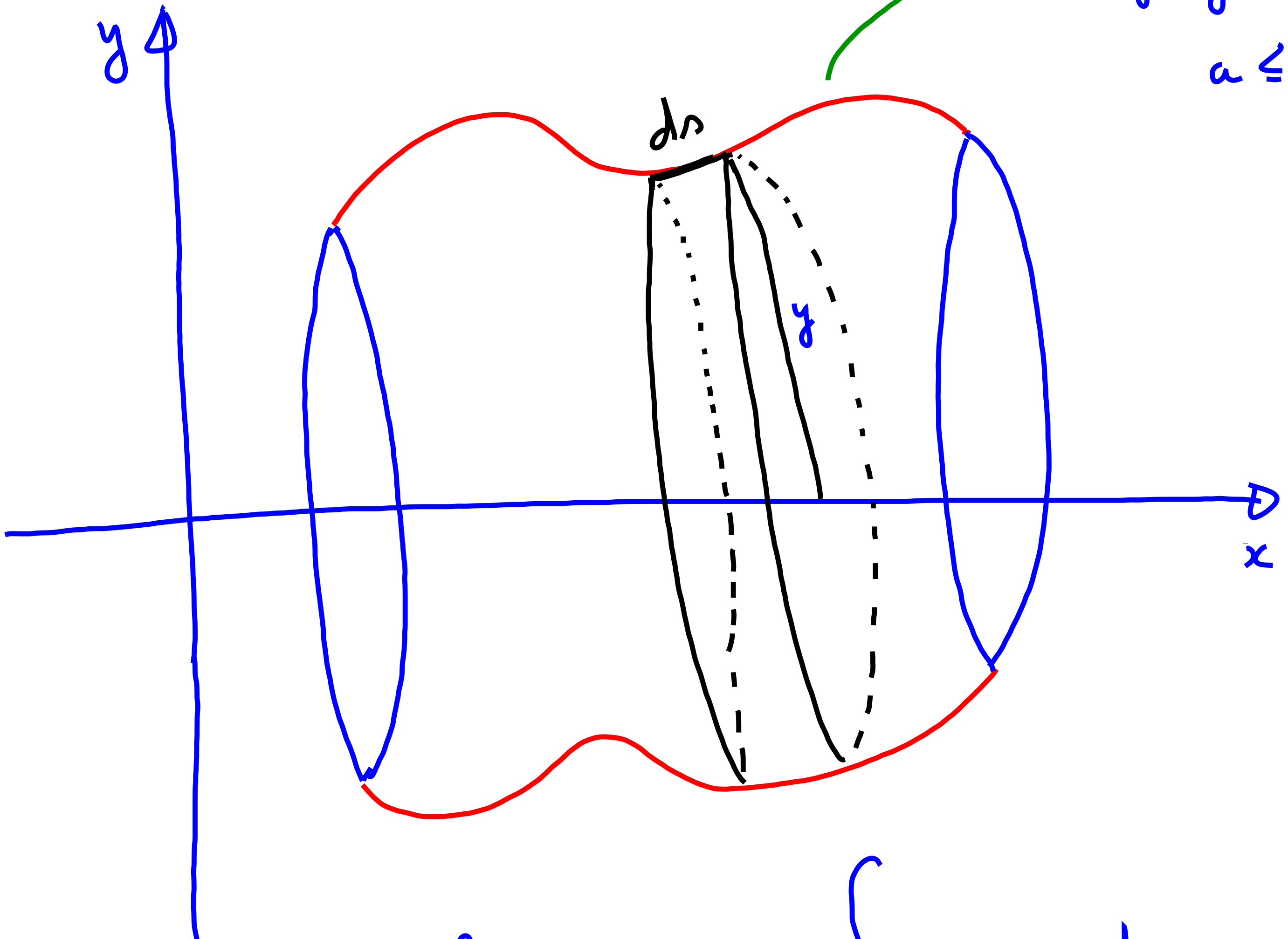
let $u = 1 + t^2$

$du = 2t\, dt$

$$= 3\int \sqrt{u}\; du = 3\cdot\frac{2u^{3/2}}{3} = 2u^{3/2} = 2\cdot(1+t^2)^{3/2}\Big|_1^3 =$$

$$\boxed{2\cdot\left[(10)^{3/2} - 2^{3/2}\right]}$$

Surface Area



$x = x(t)$

$y = y(t)$

$a \le t \le b$

$$\text{Surface area} = \int 2\pi y \cdot ds = \int_a^b 2\pi \underline{\underline{y}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

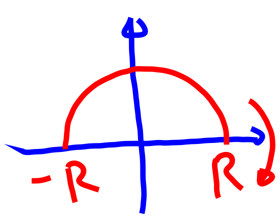
Formula for surface area of the surface obtained by rotating the curve $x = x(t)$; $y = y(t)$, $a \leq t \leq b$ about x-axis is

$$\boxed{S = 2\pi \int_a^b y(t) \cdot \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt}$$

E.g. Rotate the curve defined by (R : constant)

$$x = R\cos t \quad ; \quad y = R\sin t \quad ; \quad 0 \leq t \leq \pi .$$

about x-axis. Find surface area of resulting surface.



$x'(t) = -R\sin t; \quad y'(t) = R\cos t$

$$S = 2\pi \int_0^\pi R\sin t \cdot \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= 2\pi \int_0^\pi R\sin t \cdot \underbrace{\sqrt{R^2\sin^2 t + R^2\cos^2 t}}_{R}\, dt$$

$$= 2\pi \int_0^\pi R^2 \sin t\, dt = 2\pi R^2 (-\cos t)\Big|_0^\pi = \boxed{4\pi R^2}$$