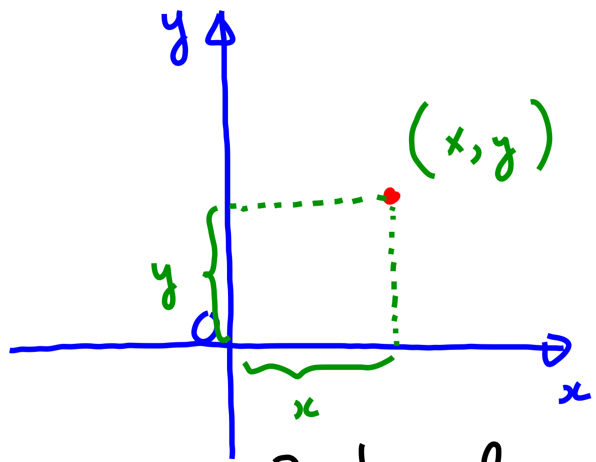
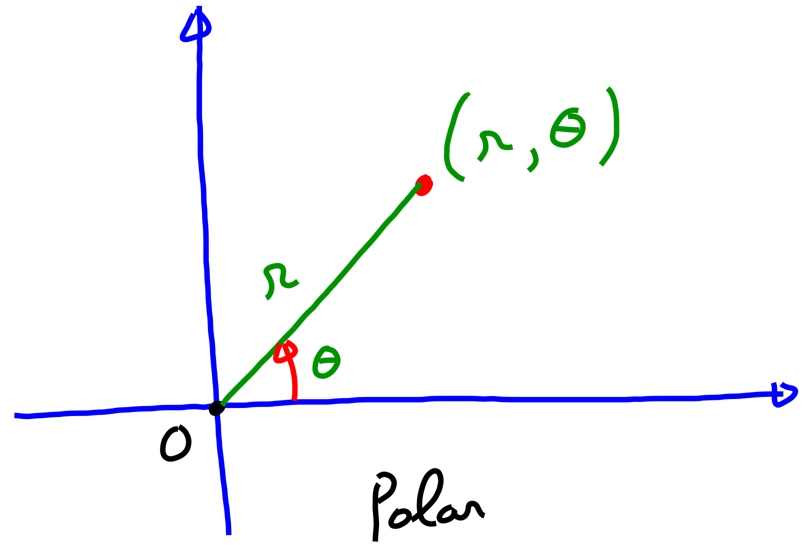
Calculus of Polar Curves

Polar Coordinates

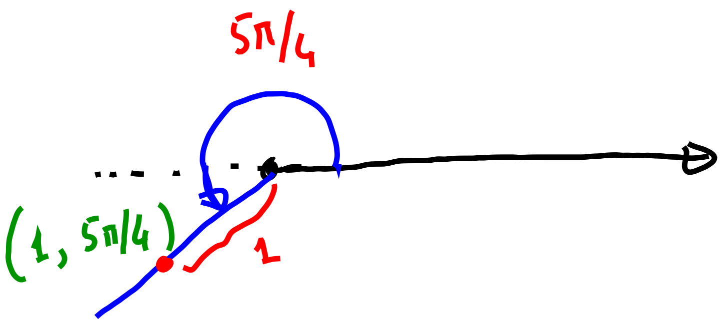


Rectangular

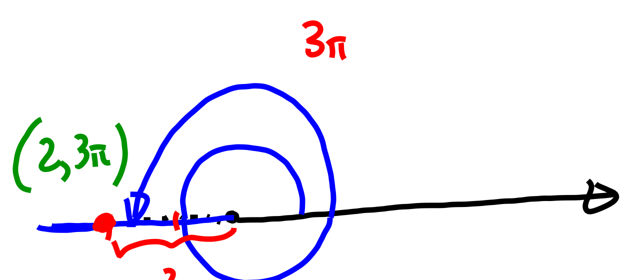


Plot the points with given polar coordinates.

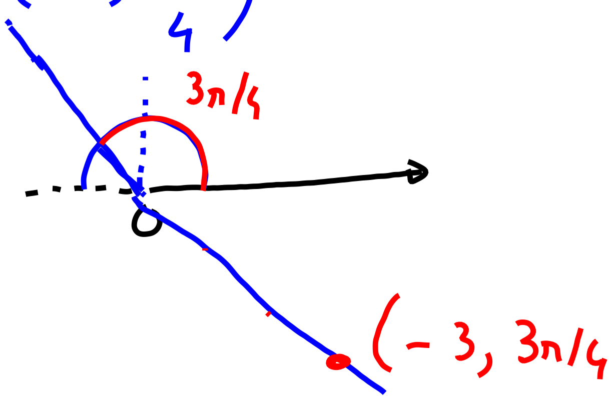
* $\left(1, \frac{5\pi}{4}\right)$

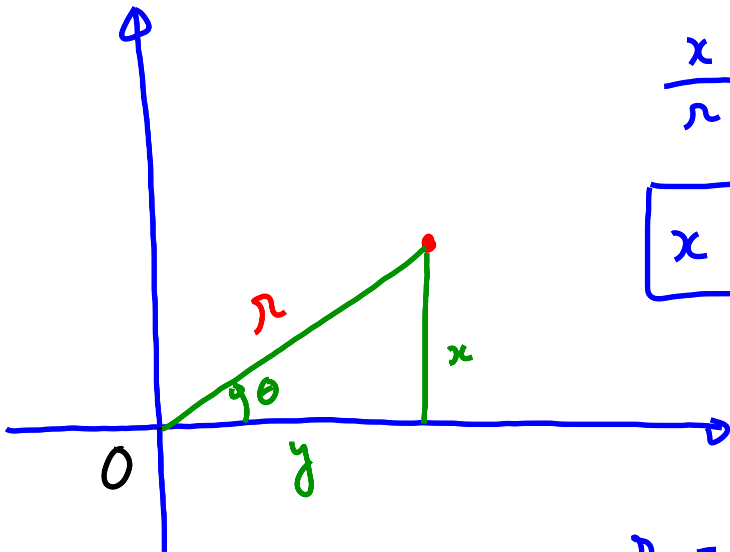


* $(2, 3\pi)$



* $\left(-3, \frac{3\pi}{4}\right)$





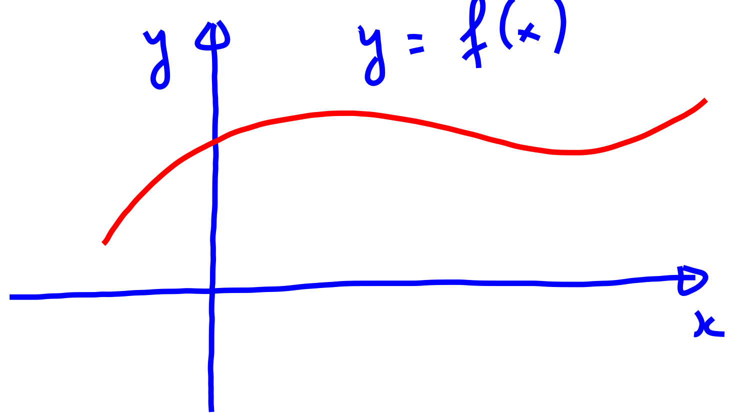
$$\frac{x}{r} = \sin\theta \qquad \frac{y}{r} = \cos\theta$$

$$\boxed{x = r\sin\theta} \qquad \boxed{y = r\cos\theta} \qquad (\text{Polar} \to \text{Rectangular})$$

$$r = \sqrt{x^2 + y^2} \; ; \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$(\text{Rectangular} \longrightarrow \text{Polar})$$

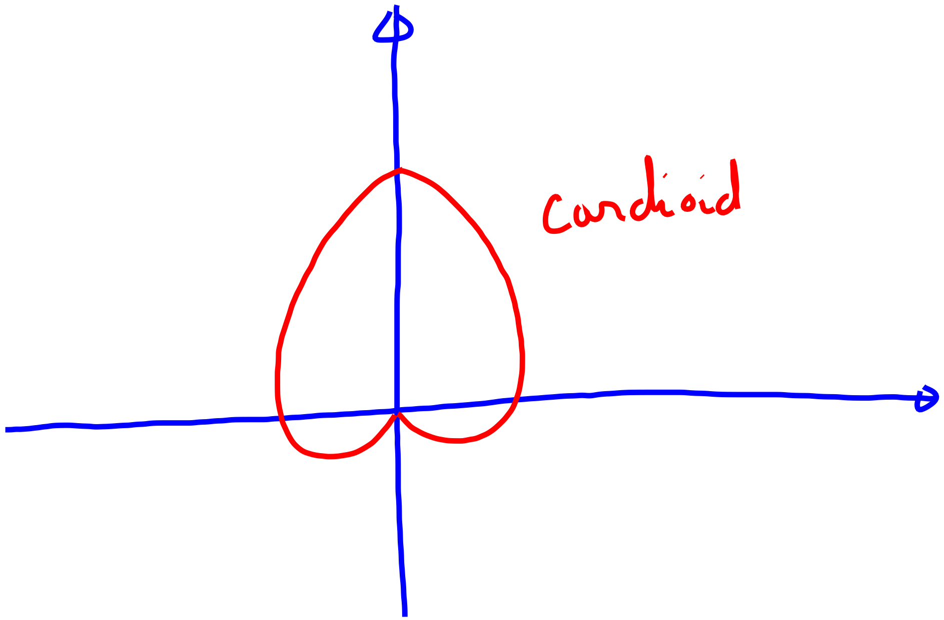
Polar Curves



A Curve in polar coordinates has the form $r = f(\theta)$

E.g. of a polar curve

$$r = 1 + \sin\theta$$



Tangents to Polar Curves

$$\text{Slope of tangent line} = \frac{dy}{dx}$$

$x = r\cos\theta$; $y = r\sin\theta$. Polar curve : $r = f(\theta)$

$x = f(\theta)\cos\theta \quad ; \quad \underbrace{y = f(\theta)\sin\theta}.$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cdot\cos\theta}{f'(\theta)\cos\theta - f(\theta)\cdot\sin\theta}$$

Recall: $r = f(\theta)$. So $f'(\theta) = \dfrac{dr}{d\theta}$.

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

E.g. Consider the Cardioid : $r = 1 + \sin\theta$.

Find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Find the equation of the tangent line at that point.

$$\text{Slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = \cos\theta + 2\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = -\sin\theta - \sin^2\theta + \cos^2\theta.$$

When $\theta = \frac{\pi}{3}$

$$\frac{dy}{d\theta} = \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$\frac{dx}{d\theta} = -\frac{\sqrt{3}}{2} - \frac{3}{4} + \frac{1}{4}$$

$$\frac{dy/d\theta}{dx/d\theta} = -1$$

Slope

Point :

$$y\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{3}{4}$$

$$x\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{4}$$

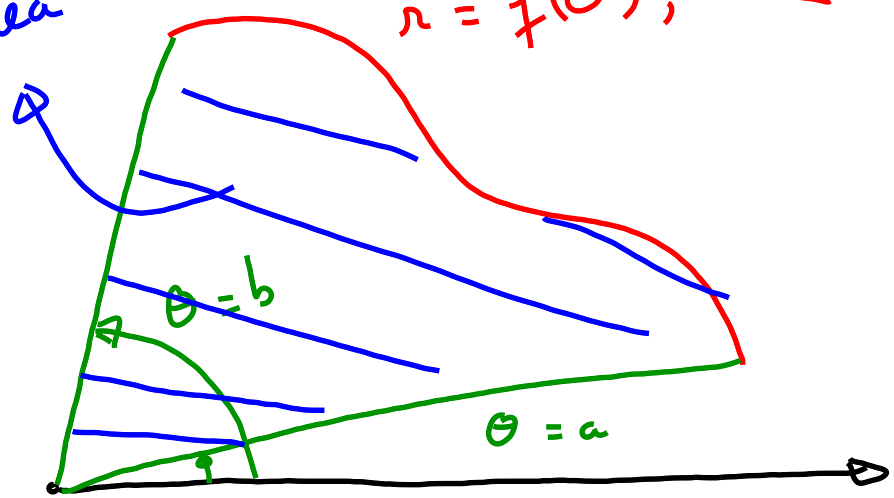
Equation of the tangent line :

$$y - \left(\frac{\sqrt{3}}{2} + \frac{3}{4}\right) = -1 \cdot \left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right)\right)$$

Area Problem with polar coordinates.

Area = ?

$r = f(\theta)$, $a \leq \theta \leq b$

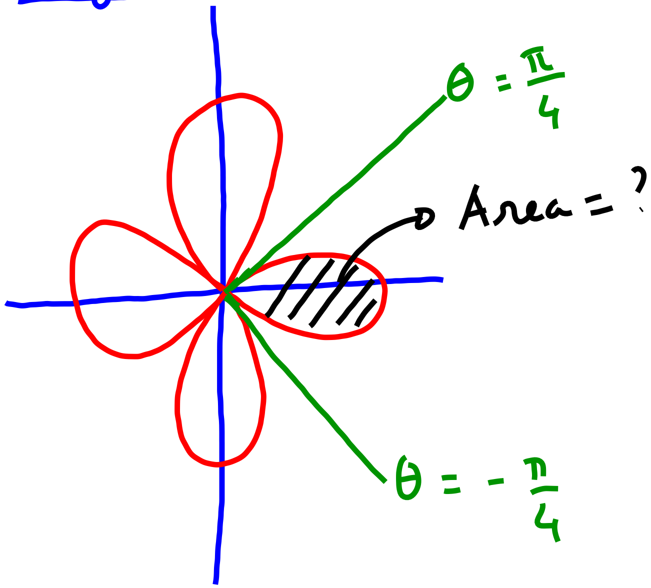


$$\boxed{\text{Area} = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 \, d\theta}$$

E.g. $r = \cos(2\theta)$

Find the area of one loop of the rose curve

$$r = \cos(2\theta) \; ; \; -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$



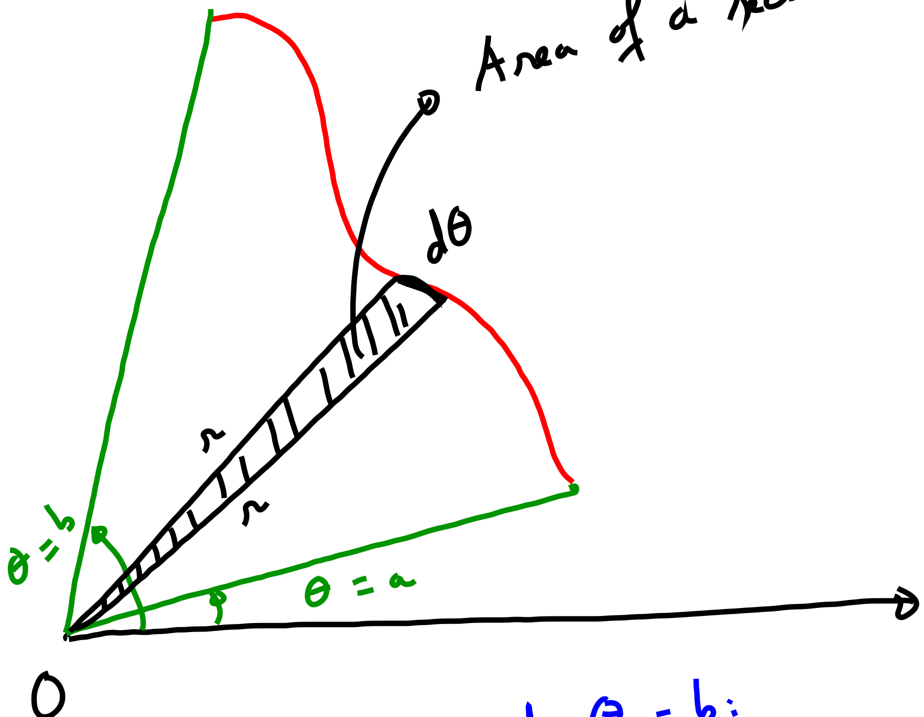
$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta$$

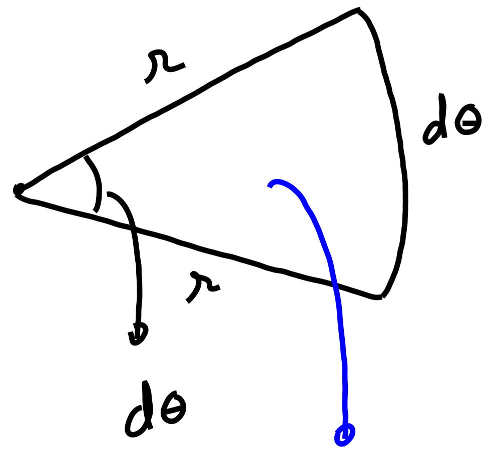
Power Reduction formula: $\cos^{\boxed{2}}(\text{Stuff}) = \dfrac{1 + \cos(2 \cdot \text{Stuff})}{2}$

$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1 + \cos(4\theta)}{2}\, d\theta = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \left[1 + \cos(4\theta)\right] d\theta = \frac{1}{4} \left(\theta + \frac{\sin(4\theta)}{4}\right) \Bigg|_{-\pi/4}^{\pi/4}$$

$$= \boxed{\frac{\pi}{8}}$$

Area of a sector = ?





Area = ?

Angle	Area
2π	πr^2
$d\theta$	?

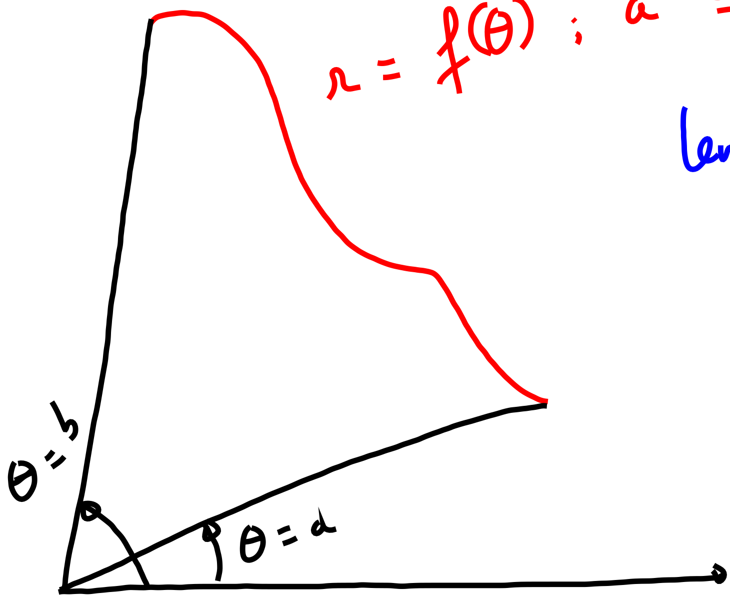
Area from $\theta = a$ to $\theta = b$:

$$\int_a^b \frac{1}{2} r^2 \, d\theta$$

Area of sector $= \dfrac{\pi r^2}{2\pi} \cdot d\theta = \dfrac{1}{2} r^2 \, d\theta$

Arc Length

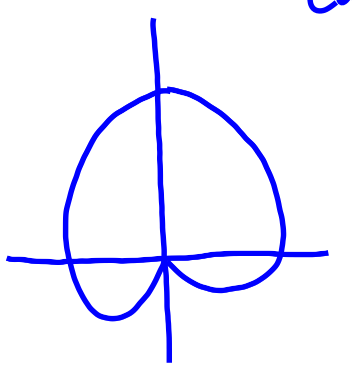
$r = f(\theta)\ ;\ a \le \theta \le b.$



$$\text{length of curve} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

E.g

Cardioid: $r = 1 + \sin\theta\ ;\ 0 \le \theta \le \frac{\pi}{2}$



Find the length of this curve.

$$L = \int_0^{\pi/2} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = \int_0^{\pi/2} \sqrt{2(1+\sin\theta)}$$

$$\sqrt{2}\int_0^{\pi/2}\sqrt{1+\sin\theta}\,d\theta = \sqrt{2}\cdot\int_0^{\pi/2}\sqrt{1+\sin\theta}\cdot\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta$$

$$= \sqrt{2}\cdot\int_0^{\pi/2}\frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\,d\theta$$

$$= \sqrt{2}\cdot\int_0^{\pi/2}\frac{\sqrt{\cos^2\theta}}{\sqrt{1-\sin\theta}}\,d\theta = -\sqrt{2}\cdot\int_0^{\pi/2}\frac{\overbrace{-\cos\theta\,d\theta}^{du}}{\sqrt{1-\sin\theta}}$$

$$u = 1-\sin\theta \;;\quad du = -\cos\theta\,d\theta$$

$$-\sqrt{2}\cdot\int\frac{du}{\sqrt{u}} = -\sqrt{2}\cdot 2\sqrt{u} = -2\sqrt{2}\cdot\sqrt{1-\sin\theta}\,\Big|_0^{\pi/2}$$

$$= \boxed{2\sqrt{2}}\,.$$