Exam 1 Review

Section 2.1. Areas Between Curves

- We can use definite integrals to find the area between two curves.
- The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the vertical lines $x = a$, $x = b$ where $f(x) \geq g(x)$ for all x in $[a, b]$ is given by the formula

$$\text{Area } = \int_a^b [f(x) - g(x)]\, dx.$$

- The area of the region bounded by the curves $x = u(y)$, $x = v(y)$ and the horizontal lines $y = c$, $y = d$ where $u(y) \geq v(y)$ for all y in $[c, d]$ is given by the formula

$$\text{Area } = \int_c^d [u(y) - v(y)]\, dy.$$

- If the graphs of the functions cross, we need to integrate the absolute value of the difference to find the area between $y = f(x)$ and $y = g(x)$, $a \leq x \leq b$

$$\text{Area } = \int_a^b |f(x) - g(x)|\, dx.$$

  To evaluate the integral with the absolute value, we must break it into two or more integrals. To break it, we must find the intersection points of the curves. These will determine the bounds of the integral(s). If we are integrating with respect to x, then we are interested in the x coordinates of the intersection points, while if we are integrating with respect to y, then we are interested in the y coordinates of the intersection points.

Section 2.2. Volumes by Slicing - Disk and Washer Method

- We can find the volume of a solid by integrating the cross-sectional area. If the cross-sectional areas of a solid S are given by a function $A(x)$, $a \leq x \leq b$ (slicing along the x-axis) then the volume of S is

$$V_S = \int_a^b A(x) dx.$$

  If the cross-sectional areas are given by $A(y)$, $c \leq y \leq d$ (slicing along the y-axis), then

$$V_S = \int_c^d A(y) dy.$$

- For solids of revolution, if the cross sections are disks, then the cross-sectional area is

$$A = \pi (\text{radius})^2.$$

If we revolve the region bounded by $y = f(x) \geq 0$, $a \leq x \leq b$ about the x-axis, then the radius of the cross-section at x is $f(x)$ and the volume of the solid obtained is

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

If we revolve the region bounded by $x = g(y) \geq 0$, $c \leq y \leq d$ about the y-axis, then the radius of the cross-section at y is $g(y)$and the volume of the solid obtained is

$$V = \pi \int_c^d [g(y)]^2 \, dx.$$

- If a solid of revolution has a cavity in the center, the cross-sections are washers and the cross-sectional area is

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

If we revolve the region bounded by $y = f(x)$, $y = g(x)$, $a \leq x \leq b$ about the x-axis where $f(x) \geq g(x)$ for all x in $[a, b]$, then the outer radius is $f(x)$, the inner radius is $g(x)$ and the volume of the solid obtained is

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2\right] dx.$$

If we revolve the region bounded by $x = u(y)$, $x = v(y)$, $c \leq y \leq d$ about the y-axis where $u(y) \geq v(y)$ for all y in $[c, d]$, then the outer radius is $u(y)$, the inner radius is $v(y)$ and the volume of the solid obtained is

$$V = \pi \int_c^d \left[(u(y))^2 - (v(y))^2\right] dy.$$

- If we revolve about a horizontal line $y = \beta$ other than the x-axis or a vertical line $x = \alpha$ other than the y-axis, then it is necessary to draw a picture to calculate the formula for the radius of a cross-section before finding the volume.

Section 2.3. Volumes of Revolution - Shell Method

- Some volumes problem are very difficult to handle by the disk or the washer method, in which case, the shell method will be very useful.

- The volume of a cylindrical shell is given by

$$V_{\text{Shell}} = (\text{circumference})(\text{height})(\text{thickness}) = 2\pi(\text{radius})(\text{height})(\text{thickness}).$$

- If we revolve the region under $y = f(x)$, $a \leq x \leq b$ about the y-axis, then the radius of a shell corresponding to rotating a slice at x is equal to x, the height of that shell is $f(x)$ and the thickness is dx. So the volume of the solid obtained is

$$V = 2\pi \int_a^b x f(x) dx.$$

- If we revolve the region under $x = g(y)$, $c \leq y \leq d$ about the x-axis, then the volume of the solid obtained is

$$V = 2\pi \int_c^d y g(y) dy.$$

- If we revolve the region bounded by $y = f(x)$, $y = g(x)$, $a \leq x \leq b$ about the y-axis where $f(x) \geq g(x)$, then the height of a shell is $f(x) - g(x)$. So the volume of the solid obtained is

$$V = 2\pi \int_a^b x \left(f(x) - g(x)\right) dx.$$

- If we revolve about an axis different from the x or y-axis, then it is necessary that we draw a picture to calculate the formula for the radius of a shell correctly.

Section 2.4. Arc Length and Surface Area

- We can find the length of a curve by dividing the curve into a large number of small line segments and integrating the line element ds.
- The line element ds is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

- The length of the curve $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

- The length of the curve $x = g(y)$, $c \leq y \leq d$ is

$$L = \int_c^d ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

- We can find the surface area of a surface of revolution by dividing the surface into a large number of small bands and integrating the band area. The area of a band is given by

$$\text{Band Area } = (\text{circumference})(\text{slant height}) = 2\pi(\text{radius})(\text{slant height}).$$

- If we revolve the curve $y = f(x)$, $a \leq x \leq b$ about the x-axis, then the radius of a band at x is $y = f(x)$ and the slant height of the band is ds. So the surface area of the surface obtained is

$$S = 2\pi \int_a^b y ds = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx.$$

- If we revolve the curve $x = g(y)$, $c \le y \le d$ about the y-axis, then the radius of a band at y is $x = g(y)$ and the slant height of the band is ds. So the surface area of the surface obtained is

$$S = 2\pi \int_c^d x ds = 2\pi \int_a^b g(y)\sqrt{1 + [g'(y)]^2} dy.$$

Section 2.5. Physical Applications

- If $\rho(x)$ is the density at the point x of a thin-rod oriented along the x-axis, $a \le x \le b$, then the mass of the rod is given by

$$m = \int_a^b \rho(x) dx.$$

- If $\rho(x)$ is the radical density of a disk of radius r, then the mass of the disk is given by

$$m = 2\pi \int_0^r x\rho(x) dx.$$

- If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point $x = a$ to point $x = b$, then the work done on the object is

$$W = \int_a^b F(x) dx.$$

- The strategy to calculate the work done to pump an amount of water (or liquid) out of a tank is as follows
  - Calculate the volume of a representative layer of water (or liquid). If the cross-sectional area of a layer at a specific height h is $A(h)$, then this volume is given by

    $$\text{Volume of a layer } = \text{(cross-sectional area)(thickness)} = A(h)dh.$$

  - Multiply the volume by ω the weight-density of water (or liquid) to get the force. Multiply the force by the distance this layer of water (or liquid) must be lifted to get the work done in lifting this layer. If the distance is $D(h)$, then the work done in lifting this layer is

    $$\text{Work to lift a representative layer} = \omega D(h)A(h)dh.$$

  - Integrate the work done in lifting a representative layer to obtain to work done in lifting the entire body of water (or liquid), this work is given by

    $$W = \int_a^b \omega D(h)A(h)dh.$$

- The hydrostatic force exerted on an object submerged in a liquid is given by

$$F = \int_a^b \rho w(x)s(x)dx,$$

where $w(x)$ is the width function, $s(x)$ is the depth function and ρ is the weight-density of the liquid.