Exam 1 Review Problems

MATH 2414

The following problems can help you internalize the important concepts and practice applying the important formulas from section 2.1 through section 2.5 in many different situations.

Problem 1. Find the area of the region enclosed by the given curves.

(a) $y = x^2 - 2x$, $y = x + 4$.

(b) $x = 2y$, $x = y^3 - y$.

(c) $y = \tan x$, $y = 2\sin x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

(d) $y = |x|$, $y = x^2 - 2$

(e) $x = \sin y$, $x = \cos(2y)$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(f) $y = \frac{1}{x}$, $y = x$, $y = \frac{1}{4}x$, $x > 0$.

Answers for Problem 1.

(a) $\frac{125}{6}$.

(b) $\frac{9}{2}$.

(c) $2 - 2\ln(2)$

(d) $\frac{20}{3}$

(e) $\frac{3\sqrt{3}}{2}$

(f) $\ln(2)$.

Problem 2.

(a) Find the number c such that the vertical line $x = c$ divides the region under the curve by $y = \frac{1}{x^2}$, $1 \le x \le 4$ into two regions with equal area.

(b) Find the number d such that the horizontal line $y = d$ divides the region bounded by the curves by $y = x^2$ and $y = 4$ into two regions with equal area.

Answers for Problem 2.

(a) $c = \frac{8}{5}$

(b) $d = \sqrt[3]{16}$

Problem 3. Find the volume of the solid obtained by revolving the region bounded by the given curves about the given line.

(a) $y = \sqrt{4 - x^2}$, $x = 0$, $y = 0$ about the x-axis

(b) $y = \ln x$, $y = 1$, $y = 2$ about the y-axis

(c) $y = \sqrt{1 + x^2}$, $y = \sqrt{4 - x^2}$ about the x-axis

(d) $y^2 = x$, $x = 2y$ about the y-axis

(e) $x = y^2$, $x = 1 - y^2$ about $x = 3$.

(f) $y = \sin x$, $y = \cos x$, $0 \le x \le \frac{\pi}{4}$ about $y = -1$

Hint: for part (f), the double angle formula $\cos(2x) = \cos^2 x - \sin^2 x$ will be useful.

Answers for Problem 3.

(a) $\frac{16\pi}{3}$

(b) $\frac{\pi}{2}(e^4 - e^2)$

(c) $2\pi\sqrt{6}$

(d) $\frac{64\pi}{15}$

(e) $\frac{10\sqrt{2}}{3}\pi$

(f) $\pi(2\sqrt{2} - \frac{3}{2})$

Problem 4. Consider figure 1 below. Find the volume of the solid obtained from revolving the given region about the given line.

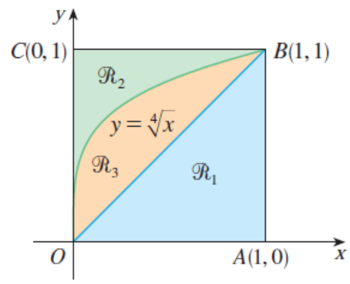

Figure 1: Figure for Problem 4

(a) $\mathcal{R}_1$ about OC

(b) $\mathcal{R}_1$ about BC

(c) $\mathcal{R}_2$ about AB

(d) $\mathcal{R}_2$ about BC

(e) $\mathcal{R}_3$ about OC

(f) $\mathcal{R}_3$ about AB

Answers for Problem 4.

(a) $\frac{2\pi}{3}$

(b) $\frac{2\pi}{3}$

(c) $\frac{13\pi}{45}$

(d) $\frac{\pi}{15}$

(e) $\frac{2\pi}{9}$

(f) $\frac{17\pi}{45}$

Problem 5. Find the volume of a sphere of radius R with a cap of height h removed from the top as in figure 2.

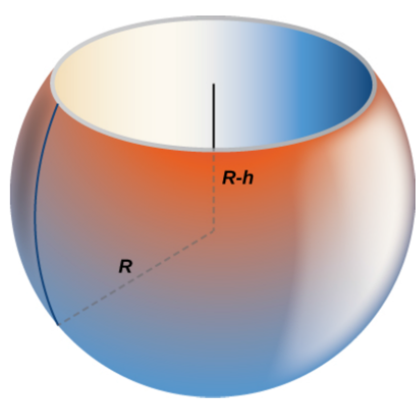

Figure 2: Figure for Problem 5

Answer for Problem 5. $\frac{\pi}{3}(R+h)(h-2R)^2$

Problem 6. Find the volume of the solid S described below by the slicing method.

(a) The base of S is the region under the parabola $y = 1 - x^2$ and above the x-axis. Cross-sections (slices) perpendicular to the y-axis are squares.

(b) The base of S is the triangular region with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. Cross-sections (slices) perpendicular to the y-axis are equilateral triangle.

(c) The base of S is the region between $y = x$ and $y = x^2$. Cross-sections (slices) perpendicular to the x-axis are semicircles.

Answers for Problem 6.

(a) 2 (b) $\frac{\sqrt{3}}{12}$ (c) $\frac{\pi}{240}$

Problem 7. Set up (but do not evaluate) the integral to find the volume of a solid torus, i.e., the donut-shaped solid with radii r and R in figure 3 below.

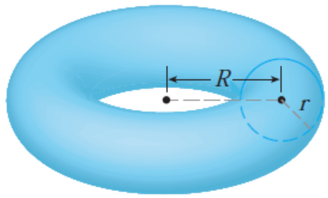

Figure 3: Figure for Problem 7

Answer for Problem 7. $8\pi R \int_0^r \sqrt{r^2 - y^2} dy$

Problem 8. Use the shell method to find the volume of the solid obtained by revolving the given curves about the given axis.

(a) $y = \sin(x^2)$, $x = 0$, $x = \sqrt{\pi}$ about the y-axis

(b) $xy = 1$, $x = 0$, $y = 1$, $y = 3$ about the x-axis

(c) $y = x^2$, $y = 2 - x^2$ about $x = 1$

(d) $x = (y - 3)^2$, $x = 4$ about $y = 1$

Answers for Problem 8.

(a) 2π (b) 4π (c) $\frac{16\pi}{3}$ (d) $\frac{128\pi}{3}$

Problem 9. Use the shell method to derive the following volume formulas.

(a) The volume of a sphere of radius r.

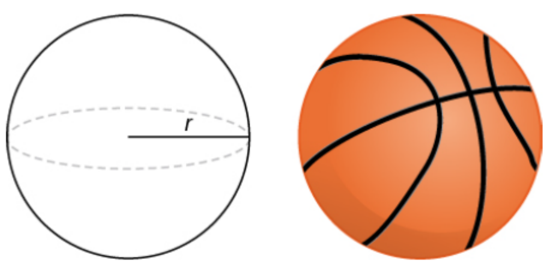

Figure 4: Figure for Problem 9a

(b) The volume of a cone with radius r and height h.

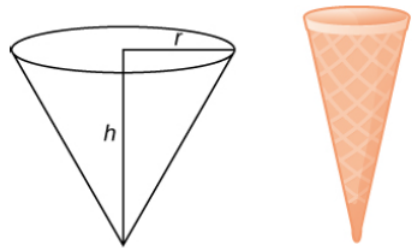

Figure 5: Figure for Problem 9b

Answers for Problem 9.

(a) $V = \frac{4}{3}\pi r^3$ (b) $V = \frac{1}{3}\pi r^2 h$

Problem 10. Find the length of the given curve over the given interval.

(a) $y = \dfrac{1}{3}(x^2+2)^{3/2}$, $0 \le x \le 1$.

(b) $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$, $\frac{1}{2} \le y \le 2$

(c) $y = \ln(\cos x)$, $0 \le x \le \frac{\pi}{4}$

Answers for Problem 10.

(a) $\frac{4}{3}$ (b) $\frac{33}{16}$ (c) $\ln(1+\sqrt{2})$

Problem 11. Find the surface area of the surface obtained by revolving the given curve about the given axis.

(a) $y = x^2$, $0 \le x \le \sqrt{2}$ about the x-axis

(b) $y = \sqrt{x}$, $2 \le x \le 6$ about the x-axis

(c) $x = \sqrt{1+e^y}$, $0 \le y \le 1$ about the y-axis

Answers for Problem 11.

(a) $\frac{13\pi}{3}$ (b) $\frac{49\pi}{3}$ (c) $\pi(e+1)$

Problem 12.

(a) Find the mass of a pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = \frac{5}{x}$ oz/in.

(b) Find the mass of a disk of radius 4 in. with radical density function $\rho(x) = \sqrt{x}$.

Answers for Problem 12.

(a) $\ln(243)$ (b) $\frac{128\pi}{5}$

Problem 13.

(a) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

(b) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

Answers for Problem 13.

(a) 6 J (b) 5 cm

Problem 14. Each tank in figure 6 below is full of water. Find the work required to pump the water out of the spout. Water density is 1000kg/m^3 or 62.5lb/ft^3 depending on the unit used in the problem.

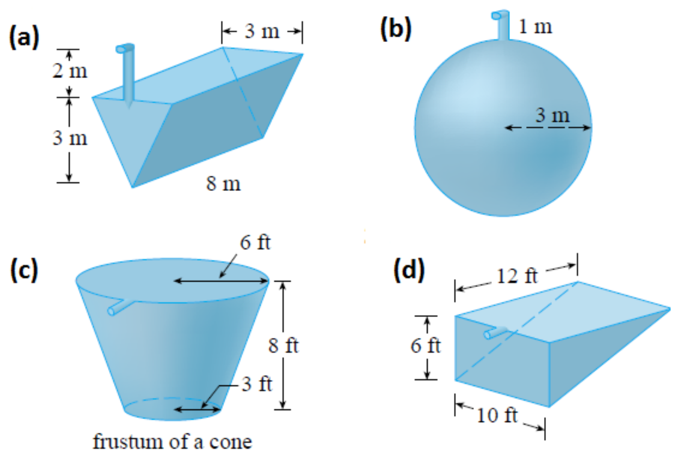

Figure 6: Figure for Problem 14

Answers for Problem 14.

(a) 1.06×10^6 J (b) 1411200π J (c) 33000π ft-lb (d) 45000 ft-lb

Problem 15. A vertical plate is submerged in water and has the given shape. Set up and evaluate the integral to find the hydrostatic force against one side of the plate. Water density is 1000kg/m^3 or 62.5lb/ft^3 depending on the unit used in the problem. In the last problem, use δ for water density instead of using an actual number. You may need an antiderivative that involves an inverse trig function in part (c)

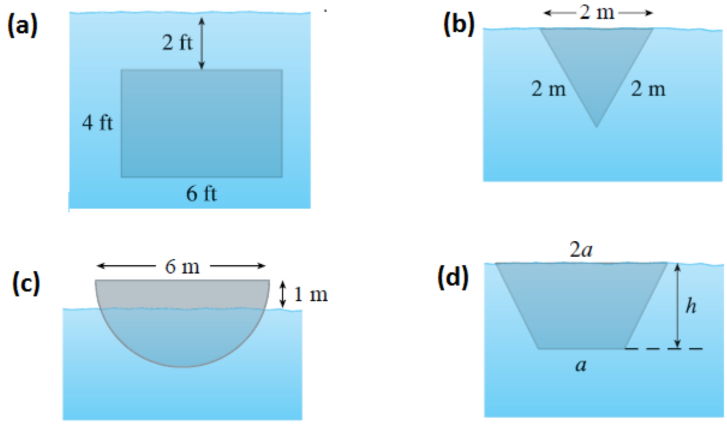

Figure 7: Figure for Problem 15

Answers for Problem 15.

(a) 6000 lb (b) 9.8×10^3 N (c) 6.7×10^4 N (d) $\frac{2}{3}\delta a h^2$