Exam 2 Review Math 2414

Section 2.6. Moments and Centers of Mass

- We can find the center of mass of a system of point masses $m_1, \ldots, m_n$ distributed along a number line at coordinates $x_1, \ldots, x_n$. The formula for the center of mass $\overline{x}$ is
$$\overline{x} = \frac{M}{m},$$
where $M = \sum_{i=1}^{n} m_i x_i$ is the moment of the system and $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

- For point masses distributed in a plane at coordinates $(x_1, y_1), \ldots, (x_n, y_n)$, the center of mass is a point with coordinates $(\overline{x}, \overline{y})$ where $\overline{x}$ and $\overline{y}$ are given by the formulas
$$\overline{x} = \frac{M_y}{m}, \overline{y} = \frac{M_x}{m},$$
here $M_y = \sum_{i=1}^{n} m_i x_i$ is the moment of the system with respect to the y-axis and $M_x = \sum_{i=1}^{n} m_i y_i$ is the moment of the system with respect to the x-axis.

- For a lamina (thin plate) with uniform density ρ bounded by the graphs of two functions $g(x) \leq y \leq f(x)$, $a \leq x \leq b$, the moments are given by
$$\text{Moment about } y\text{-axis: } M_y = \rho \int_a^b x\,(f(x) - g(x))\,dx,$$
$$\text{Moment about } x\text{-axis: } M_x = \rho \int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right)dx.$$
The total mass is given by
$$m = \rho \int_a^b (f(x) - g(x))\,dx.$$
And the *centroid* (center of mass) $(\overline{x}, \overline{y})$ is given by $\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$ and the ρ will be canceled when you divide.

- The symmetry principle can make our lives much easier if recognized and applied correctly. The centroid will always lie on a line of symmetry if the lamina has one. For example, if the lamina is bounded by the functions $y = x^2$, $y = 0$, $-1 \leq x \leq 1$, then the centroid will belong to the y-axis, hence, $\overline{x} = 0$. If the lamina is a circle, then the centroid will be the center. If the lamina is a rectangle, then the centroid will be the point of intersection of the two diagonals, and so on.

- We can use the Theorem of Pappus to calculate volumes: if we revolve a region $\mathcal{R}$ about a line l, then the volume of the solid obtained is given by
$$\text{Volume } = (\text{Area of } R) \times (\text{d}),$$

where d is the distance traveled by the centroid of R. More explicitly, since the path travelled by the centroid is a circle, the above formula can be rewritten as

$$\text{Volume } = \underbrace{2\pi \times (\text{distance from the centroid to rotation axis})}_{d} \times (\text{Area of } R).$$

Section 3.1. Integration by Parts

- The method of integration by parts allows the exchange of one integral for another, possibly easier, integral.
- The formula for integration by parts is

$$\int u dv = uv - \int v du.$$

- If we have a definite integral, then the formula is

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

- The method of integration by parts is often used to integrate functions of the form $x^n \sin(x)$, $x^n \cos(x)$, $x^n e^x$, $x^n \ln x$, $e^{nx} \sin(mx)$, etc. To apply integration by parts, we normally want to choose u to be a function that will become simpler when differentiated and choose dv to be something that can be readily integrated.
- The acronym **LIATE** can often help to take some of the guess work out of our choices for u and dv. This acronym stands for **L**ogarithmic functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric functions, **E**xponential Functions. The type of function that appears first in the list should be our choice for u. For example, the integral $\int x^n \sin(x) dx$ has an **A**lgebraic Function x^n and a **T**rigonometric Function $\sin(x)$. Because **A** comes before **T** in **LIATE**, we choose $u = x^n$ and $dv = \sin(x)dx$.
- Integration by parts is also useful for integrating inverse functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\ln(x)$ or functions involving these as factors. In these cases, we should choose $u = \sin^{-1} x$, or $u = \cos^{-1} x$ or $u = \tan^{-1} x$ or $u = \ln(x)$, even if there are no other factors in the integrand, i.e., we can choose $dv = dx$.

Section 3.2. Trigonometric Integrals

- For $\displaystyle\int \sin^n(x) \cos^m(x) dx$
  - If m is odd, we save a cosine factor, use the identity $\cos^2(x) = 1 - \sin^2(x)$ to convert the rest of the integrand to a function of sine only and make the substitution $u = \sin(x)$.
  - If n is odd, we save a sine factor, use the identity $\sin^2(x) = 1 - \cos^2(x)$ to convert the rest of the integrand to a function of cosine only and make the substitution $u = \cos(x)$.
  - If both m and n are even, we use the power reduction formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \text{ and } \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

    to change the integrand into something we can integrate.

- For $\int \tan^n(x)\sec^m(x)dx$
  - If m is even, we save a factor of $\sec^2(x)$, use the identity $\sec^2(x) = \tan^2(x)+1$ to convert the rest of the integrand to a function of tangent only and make the substitution $u = \tan(x)$.
  - If n is odd, we save a factor of $\sec(x)\tan(x)$, use the identity $\tan^2(x) = \sec^2(x) - 1$ to convert the rest of the integrand to a function of secant only and make the substitution $u = \sec(x)$.
  - For $\int \tan^n(x)dx$, convert one $\tan^2 x = \sec^2 x - 1$ and split the problem into two integrals.
  - The following formulas are very helpful

$$\int \tan(x)dx = \ln|\sec(x)| + C \text{ and } \int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- For $\int \sin(nx)\cos(mx)dx$, $\int \sin(nx)\sin(mx)dx$, $\int \cos(nx)\cos(mx)dx$. Use the **product-to-sum** formulas.

Section 3.3. Trigonometric Substitution

- If the integrand involves the expression $\sqrt{a^2 - b^2x^2}$, use the trig substitution $x = \frac{a}{b}\sin\theta$, then $dx = \frac{a}{b}\cos\theta d\theta$, and $\sqrt{a^2 - b^2x^2} = a\cos\theta$ and you can convert the integrand to a trig function of θ. When you finish integrating the function, use geometry or trig identities to express the answer in terms of x.
- If the integrand involves the expression $\sqrt{a^2 + b^2x^2}$, use the trig substitution $x = \frac{a}{b}\tan\theta$, then $dx = \frac{a}{b}\sec^2\theta d\theta$ and $\sqrt{a^2 + b^2x^2} = a\sec\theta$.
- If the integrand involves the expression $\sqrt{b^2x^2 - a^2}$, use the trig substitution $x = \frac{a}{b}\sec\theta$, then $dx = \frac{a}{b}\sec(\theta)\tan(\theta)d\theta$ and $\sqrt{b^2x^2 - a^2} = a\tan\theta$.
- If the integrand involves $\sqrt{ax^2 + bx + c}$, use the method of completing the square to get it in the form $\sqrt{a(x-h)^2 + k}$. Factor out the a and apply the substitution $u = x - h$ will take us to one of the three forms above.
- Avoid using trig substitution when a regular u-substitution is possible.

Section 3.4. Partial Fractions

- A rational function is a fraction of polynomials. Specifically, it is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials. The method of Partial Fractions Decompostion can be used to break down a rational function into a sum of simpler rational functions that can be integrated using previous techniques.
- Before applying partial fractions decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division first and then perform the decomposition.

- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors and repeated irreducible quadratic factors.

  - Non repeated linear factors in the denominator $Q(x)$:

    $$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_mx + b_m).$$

    In this case, the decomposition takes the form

    $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_m}{a_mx + b_m}.$$

    Then we must determine the contants $A_1, A_2, \dots A_m$ and integrate the right hand side.

  - If $Q(x)$ contains a repeated linear factor, say the factor (a_1x+b_1) get repeated r times: that is $(a_1x + b_1)^r$ occurs in $Q(x)$. Then instead of the single term $\frac{A_1}{a_1x+b_1}$ in the right hand side, we will have

    $$\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \dots + \frac{B_r}{(a_1x + b_1)^r}.$$

  - Nonrepeated irreducible factors: If $Q(x)$ contains an irreducible quadractic factor $ax^2 + bx + c$ (irreducible just means you cannot factor it over the real numbers) and it is not repeated, then the right hand side of the decomposition will have a term of the form

    $$\frac{Ax + B}{ax^2 + bx + c}.$$

    To integrate this term, we need to complete the square in the denominator and turn it into the form

    $$\int \frac{Cu + D}{u^2 + k^2} = C\int \frac{u}{u^2 + k^2} + D\int \frac{1}{u^2 + k^2}.$$

    We can integrate the first term in the right hand side by a u-substitution and integrating the second term results in an inverse trig function.

  - If $Q(x)$ contains a repeated irreducible quadratic factor $(ax^2 + bx + c)^r$, then instead of the single term $\frac{Ax+B}{ax^2+bx+c}$ in the decomposition, we will have

    $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots \frac{A_1x + B_1}{(ax^2 + bx + c)^r}.$$