Exam 3 Review

Section 3.6. Numerical Integrations

- This section addresses two important problems: how can we estimate the value of a definite integral when the closed form of the antiderivative is hard to find and how can we approximate the error of our estimation?

- Three commonly used methods for numerical integration are the **midpoint rule**, the **trapezoidal rule** and the **Simpson's rule**.

- The **midpoint rule** approximates definite integrals using rectangular regions. To approximate a definite integral $\int_a^b f(x)dx$ using the midpoint rule, we divide the interval $[a, b]$ into n subintervals, each of which has the same length $\Delta x = \frac{b-a}{n}$. The endpoints of these subintervals are $x_0 = a$, $x_1 = x_0 + \Delta x, \ldots, x_i = x_0 + i\Delta x, \ldots, x_n = b$. Let $m_i = \frac{x_{i-1} + x_i}{2}$ be the midpoint of the i^{th} subinterval $[x_{i-1}, x_i]$. The formula for the midpoint rule is

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x.$$

- The **trapezoidal rule** approximates definite integrals using trapezoids rather than rectangles. The formula for the trapezoidal rule is

$$T_n = \frac{\Delta x}{2}\left(f(\underbrace{x_0}_{a}) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(\underbrace{x_n}_{b}))\right).$$

  Using summation notation, the formula is

$$T_n = \frac{\Delta x}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right).$$

- **Simpson's rule** approximates definite integrals using areas under parabolas rather than areas of trapezoids or rectangles. The formula for Simpson's rule is

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right).$$

  Keep in mind that n must be **even** to use Simpson's rule.

- In practice, if we estimate the value of a definite integral using one of the above techniques, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an **upper bound** for the error in an approximation of an integral. The following formulas provide **error bounds** for each of the rules.

- Error bound (error estimate) for the midpoint rule

$$|E_{M_n}| = \text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2},$$

where $|f''(x)| \leq M$ on $[a,b]$. In other words, M can be taken to be the maximum value of $|f''(x)|$ over the interval $[a,b]$.

- Error bound (error estimate) for the trapezoid rule

$$|E_{T_n}| = \text{Error in } T_n \leq \frac{M(b-a)^3}{12n^2},$$

where $|f''(x)| \leq M$ on $[a,b]$.

- Error bound (error estimate) for the Simpson's rule

$$|E_{S_n}| = \text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4},$$

where $|f^{(4)}(x)| \leq M$ on $[a,b]$. In other words, M can be taken to be the maximum value of $|f^{(4)}(x)|$, the absolute value of the fourth derivative of x, over the interval $[a,b]$.

You will encounter two types of problem that involve these formulas for error bounds: to estimate the error of the calculation for a particular value of n, or to find a value for n that gives an error no more than some stated value.

Section 3.7. Improper Integrals

- There are two types of **improper integrals**.
  - Integrals of continuous functions over **infinite intervals**:

$$\int_a^\infty f(x)dx, \int_{-\infty}^b f(x)dx, \int_{-\infty}^\infty f(x)dx.$$

  - Integrals of functions over an interval for which the function has an **infinite discontinuity at an endpoint**.

$$\int_a^b f(x)dx, \int_c^a f(x)dx,$$

  where f has an infinite discontinuity at a.

- To calculate an improper integral, we must turn it into a limit.

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx.$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx.$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = \lim_{t\to-\infty}\int_{t}^{0} f(x)dx + \lim_{t\to\infty}\int_{0}^{t} f(x)dx.$$

If f has an infinite discontinuity at the endpoint a, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to a^+}\int_{t}^{b} f(x)dx.$$

$$\int_{c}^{a} f(x)dx = \lim_{t\to a^-}\int_{c}^{t} f(x)dx.$$

- Watch out for infinite discontinuities in the middle of the interval. You must split the integral at the discontinuities in that case. For example, if we want to integrate $\int_{-1}^{1} \frac{1}{x^3}dx$, we have to split the integral at 0 because the integrand $\frac{1}{x^3}$ is discontinuous at $x = 0$. So,

$$\int_{-1}^{1} \frac{1}{x^3}dx = \int_{-1}^{0} \frac{1}{x^3}dx + \int_{0}^{1} \frac{1}{x^3}dx.$$

  Now we have two improper integrals of the second type that we need to deal with. Note that

$$\int_{-1}^{0} \frac{1}{x^3}dx = \lim_{t\to 0^-}\int_{-1}^{t} \frac{1}{x^3}dx = \lim_{t\to 0^-}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \infty.$$

  This is sufficient to conclude that the original integral diverges. If we had not noticed the discontinuity at $x = 0$ and evaluate the original integral "as usual," we would have gotten $\int_{-1}^{1} \frac{1}{x^3}dx = 0$ which is incorrect.

- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

- A very useful result about improper integral to keep in mind is

$$\int_{1}^{\infty} \frac{1}{x^p}dx \text{ is convergent if } p > 1 \text{ and is divergent if } p \le 1.$$

Section 5.1. Sequences

- An infinite sequence $\{a_n\}$ is an ordered list of numbers $a_1, a_2, a_3, \ldots$.

- We say that a sequence **coverges** to a real number L if we can make the terms a_n as close to L as we like by taking n sufficiently large. In this case, we write $\lim_{n\to\infty} a_n = L$ or $a_n \longrightarrow L$ and say that the sequence $\{a_n\}$ is a **convergent sequence**. Otherwise, we say that $\{a_n\}$ diverges or $\{a_n\}$ is a **divergent sequence**.

- Many sequences we deal with are given by an explicit formula for the n^{th} term, $a_n = f(n)$. For such sequences, if $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence is $\lim_{n\to\infty} a_n = L$ as well. This observation is useful because we can use everything we know about limits of functions to find limits of sequences. And of the many ways to find limits of functions, L'Hopital rule is an effective tool, so you should expect to use it.

- We can also find limits of sequences by using the Squeeze Theorem: If $a_n \leq b_n \leq c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ for some real number L, then $\lim_{n\to\infty} b_n = L$.

- A sequence is bounded above (resp. below) if there is a number M such that $a_n \leq M$ (resp. $a_n \geq M$) for all n. We say that a sequence is bounded if it is bounded above and below.

- A sequence is eventually increasing (resp. decreasing) if $a_n \leq a_{n+1}$ (resp. $a_n \geq a_{n+1}$) for all $n \geq N_0$. We say that a sequence is **monotonic** if it is either (eventually) increasing or decreasing. Note that if a sequence is given by an explicit formula $a_n = f(n)$, we can check whether it is evantually increasing or decreasing by considering the first derivative $f'(n)$.

- Every bounded, monotonic sequence is convergent.

- It is useful to keep in mind the following limits:

  - If r is a real number and $-1 < r < 1$, then $r^n \longrightarrow 0$. If $r = 1$, then $r^n \longrightarrow 1$, obviously. If $r > 1$ or $r \leq -1$, then the sequence r^n is divergent.
  - If c is any real, positive number, then $\dfrac{1}{n^c} \longrightarrow 0$.
  - $n^{\frac{1}{n}} \longrightarrow 1$.
  - $\left(1 + \dfrac{c}{n}\right)^n \longrightarrow e^c$.

  The third and fourth limits are obtained by using L'Hopital rule.

Section 5.2. Infinite Series

- An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots.$$

The numbers $a_1, a_2, \dots$ are called the terms of the series. The value of an infinite series is defined in terms of the limit of the sequence of partial sums $\{S_k\}$ where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k,$$

S_k is called the k^{th} partial sum of the series, it is the sum of the first k terms of the series.

The series converges if the sequence $\{S_k\}$ converges, in which case, the sum of the series is defined to be

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k.$$

There are some particular types of series that we should keep in mind.

- The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1}$ converges if the common ratio r satisfies $|r| < 1$. It diverges if $|r| \geq 1$. For $|r| < 1$, the sum of the series is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

In words, for a convergent geometric series, the sum of the series is $\dfrac{\text{first term}}{1 - \text{common ratio}}$.

For example, the series $\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1}$ is a geometric series with common ratio $r = -\frac{2}{3}$ and the first term is 5. Hence, it converges to $\dfrac{5}{1-\left(-\frac{2}{3}\right)} = 3$.

The series $\sum_{n=1}^{\infty} e^{2n} = \sum_{n=1}^{\infty} e^2(e^2)^{n-1}$ is a geometric series with common ratio $r = e^2 > 1$. Hence, it diverges.

- A series of the form

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \ldots + [b_n - b_{n+1}] + \ldots$$

is called a **telescoping series**. The k^{th} partial sum of this series is given by

$$S_k = \sum_{n=1}^{k} [b_n - b_{n+1}] = b_1 - b_{k+1}.$$

The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists (as a finite number). In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = \lim_{k\to\infty} S_k = \lim_{k\to\infty} (b_1 - b_{k+1}) = b_1 - \lim_{k\to\infty} b_{k+1}.$$

For example, the series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$ is a telescoping series. The k^{th} partial sum is

$$S_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{k+1}.$$

Hence, the sum of the series is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1.$$

We have to be careful, not every telescoping series converges. For example, the series $\sum_{n=1}^{\infty} \ln \dfrac{n}{n+1} = \sum_{n=1}^{\infty} [\ln(n) - \ln(n+1)]$ is a telescoping series. The k^{th} partial sum is

$$S_k = \sum_{n=1}^{k} [\ln(n) - \ln(n+1)] = \ln(1) - \ln(k+1) = -\ln(k+1).$$

Since $\lim_{k\to\infty} S_k = \lim_{k\to\infty} (-\ln(k+1)) = -\infty$, the series diverges.

- The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

diverges.

Section 5.3. The Divergence and Integral Tests

- If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, the series $\sum_{n=1}^{\infty} a_n$ diverges. This is called the **Divergence Test**. This is an excellent test to start with because the limit is often easy to calculate.

  Warning: if the limit is zero, the Divergence Test tells you nothing. The series may converge or diverge. You must try some other test.

  For example, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges because the limit of the terms is $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n^2} = e^0 = 1$. On the other hand, for the series $\sum_{n=1}^{\infty} \frac{1}{n}$, the limit of the terms is $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. This does not imply that the series converges. As a matter of fact, we have seen that this is the harmonic series and it diverges.

- **The Integral Test**: If $\sum_{n=1}^{\infty} a_n$ is a series with $a_n = f(n)$ and f is a function which satisfies all of the following requirements:

  - f is continuous on $[1, \infty)$
  - f is positive on $[1, \infty)$
  - f is decreasing on $[1, \infty)$

  then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ either both converge or both diverge. In other words, if we know that $\int_1^{\infty} f(x)dx$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. If $\int_1^{\infty} f(x)dx$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (Note: the interval does not have to be strictly $[1, \infty)$, as long as $a_n = f(n)$ for $n \geq N$ and f satisfies all the above requirements on the interval $[N, \infty)$ for some natural number N, the integral test will still apply.)

The integral test is handy if the function associated with the series can be integrated without too much difficulty.

- If the integral test can be applied, we can also estimate the remainder (the tail) of the series. More specifically, suppose that we use the N^{th} partial sum $S_N = \sum_{n=1}^{N} a_n$ to estimate the sum of the infinite series $\sum_{n=1}^{\infty} a_n$. Then S_N is accurate up to an error $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$. The integral test can help us find an estimate for R_N:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx.$$

For example, suppose we want to use the 5^{th} partial sum S_5 to approximate the convergent infinite series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$. We have

$$S_5 = \sum_{n=1}^{5} \frac{1}{(2n+1)^3} = \frac{1}{27} + \frac{1}{125} + \frac{1}{343} + \frac{1}{729} + \frac{1}{1331} \approx 0.0050076.$$

How close is this approximation? Of course, we don't know the exact error because if the sum of the infinite series is easy to calculate, there would be no need for the approximation. By the above formula, the error R_5 is bounded in between the following 2 numbers

$$\int_5^{\infty} \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty} \int_5^{t} \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty} \left(\frac{1}{484} - \frac{1}{4(2t+1)^2} \right) = \frac{1}{484} \approx 0.002066,$$

and

$$\int_4^{\infty} \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty} \int_4^{t} \frac{1}{(2x+1)^3}dx = \lim_{t\to\infty} \left(\frac{1}{324} - \frac{1}{4(2t+1)^2} \right) = \frac{1}{324} \approx 0.003086,$$

that is, $0.002066 < R_5 < 0.003086$.

- You will encounter two types of problem that involve these formulas for error bounds: to find the error bounds of the calculation for a particular value of n, or to find the least value for n that gives an error no more than some stated value.

- We can use the integral test to show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and it is divergent if $p \leq 1$. This is a very useful result to keep in mind.

Section 5.4. Comparison Tests

- We can use the **comparison test** to determine the convergence and divergence for many series by comparing them with geometric series or p-series, or series that we know exactly when they converge or diverge. The key here is that the series must have positive terms and if $0 \leq a_n \leq b_n$ for all $n \geq N$, then

  - If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

- The comparison test works very well if we can find a comparable series satisfying the requirement of the test. However, sometimes it may be difficult to find a series whose terms are all less than or greater than the series we deal with. In which case, if we can just find a series which "behaves like" the series we deal with, we can use the limit comparison test. We just need to find the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If the limit is **finite** and **positive**, then both series converge or both diverge. Since we already know about one of them, you then know about the other.

Section 5.5. Altenating Series Test

- An **alternating series** is a series whose terms alternate between positive and negative values. An alternating series has the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \dots$$

where $b_n \geq 0$.

- An alternating series of one of the above forms converges if the following two requirements are satisfied:
  - $\lim_{n\to\infty} b_n = 0$ and
  - $0 \leq b_{n+1} \leq b_n$ for all n

- If an alternating series converges and we use the N^{th} partial sum S_N to approximate the sum, then the remainder R_N is bounded by

$$|R_N| \leq b_{N+1}.$$

For example, the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges by the alternating series test. If we approximate the infinite sum by the 10^{th} partial sum S_{10}, then the remainder (error in this approximation) is bounded by

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008625,$$

that is, the error in this approximation is at most 0.008625.

- A series $\sum a_n$ converges **absolutely** if the series of absolute values $\sum |a_n|$ converges. On the other hand, a series $\sum a_n$ may converge, but the absolute values series $\sum |a_n|$ may diverge. In which case, we say that the original series $\sum a_n$ converges conditionally.

- For example, the series $\displaystyle\sum_{n=1} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test. However, the absolute values series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series which we know to be divergent. Therefore, we say that the original series $\displaystyle\sum_{n=1} \frac{(-1)^{n+1}}{n}$ converges conditionally.

  On the other hand, the series $\displaystyle\sum_{n=1} \frac{(-1)^{n+1}}{n^2}$ converges by the alternating series test as well. The absolute values series is $\displaystyle\sum_{n=1} \frac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$. Therefore, we say that the original series $\displaystyle\sum_{n=1} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

- Note that if $\sum |a_n|$ converges, then $\sum a_n$ converges.

Section 5.6. Ratio and Root Tests

- **Ratio Test:** If
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$
then the series $\sum a_n$ converges absolutely if $L < 1$ and the series diverges if $L > 1$. If $L = 1$, the test fails, that is, it does not provide any information and we have to use something else. This test is very useful for series whose terms involve factorials.

- **Root Test:** If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L,$$
then the series $\sum a_n$ converges absolutely if $L < 1$ and the series diverges if $L > 1$. If $L = 1$, the test fails. This test works really well when there are powers of n in the terms a_n.

Note: Test 3 does NOT cover section 5.6. Section 5.6 is covered in Test 4. However, it is included here because it the last section of chapter 5.