Integration Review

Basic Integration Formulas

1. $\int k\,dx = kx + C \quad (\text{any number } k)$	**12.** $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$ $= -\ln\lvert\cos x\rvert + C$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	**13.** $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$ $= -\ln\lvert\csc x\rvert + C$
3. $\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$	**14.** $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	**15.** $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$	**16.** $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	**17.** $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	**18.** $\int \tanh x\,dx = \ln(\cosh x) + C$
8. $\int \sec x \tan x\,dx = \sec x + C$	**19.** $\int \coth x\,dx = \ln\lvert\sinh x\rvert + C$
9. $\int \csc x \cot x\,dx = -\csc x + C$	**20.** $\int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + C \quad (a > 0)$
10. $\int \sec^2 x\,dx = \tan x + C$	**21.** $\int \dfrac{1}{a^2 + x^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C \quad (a \neq 0)$
11. $\int \csc^2 x\,dx = -\cot x + C$	**22.** $\int \dfrac{1}{x\sqrt{x^2 - a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left\lvert\dfrac{x}{a}\right\rvert + C \quad (a > 0)$

1. Make a Basic Substitution

Example: $\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx$

$$u = x^2 - 9x + 1$$
$$du = (2x-9)\,dx$$

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \int \frac{1}{u^{\frac{1}{2}}}\,du = \int u^{-\frac{1}{2}}du = 2u^{\frac{1}{2}} + C = 2\sqrt{x^2-9x+1} + C$$

More examples for you to complete:

1. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx, u = \sqrt{x}, 2du = \frac{1}{\sqrt{x}}\,dx$

Answer: $\boxed{2e^{\sqrt{x}} + C}$

2. $\int \frac{2x}{\sqrt{1-x^4}}\,dx, u = x^2, du = 2x\,dx$

Answer: $\boxed{\sin^{-1}(x^2) + C}$

3. $\int \frac{1}{x\cos(\ln x)}\,dx, u = \ln x, du = \frac{1}{x}\,dx$

Answer: $\boxed{\ln\left|\sec(\ln x) + \tan(\ln x)\right| + C}$

2. Complete the Square

Example: $\int \frac{1}{\sqrt{8x-x^2}}\,dx$

$$8x-x^2=-\left(x^2-8x\right)=-\left(x^2-8x+16\right)+16=16-\left(x-4\right)^2$$

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx=\int \frac{1}{\sqrt{16-\left(x-4\right)^2}}\,dx$$

$$u=x-4$$
$$du=dx$$

$$\int \frac{1}{\sqrt{16-\left(x-4\right)^2}}\,dx=\int \frac{1}{\sqrt{16-u^2}}\,du=\sin^{-1}\left(\frac{u}{4}\right)+C=\sin^{-1}\left(\frac{x-4}{4}\right)+C$$

More examples for you to complete:

1. $\int \frac{1}{x^2-2x+2}\,dx,\ x^2-2x+2=1+\left(x-1\right)^2$

Answer: $\boxed{\tan^{-1}\left(x-1\right)+C}$

2. $\int \frac{1}{\left(x+1\right)\sqrt{x^2+2x}}\,dx,\ x^2+2x=\left(x+1\right)^2-1$

Answer: $\boxed{\sec^{-1}\left|x+1\right|+C}$

3. Use a Trig. Identity

Example: $\int (\csc x - \tan x)^2\,dx$

$$\begin{aligned}(\csc x - \tan x)^2 &= \csc^2 x - 2\csc x \tan x + \tan^2 x \\ &= \csc^2 x - 2\sec x + \sec^2 x - 1\end{aligned}$$

$$\begin{aligned}\int (\csc x - \tan x)^2\,dx &= \int \csc^2 x\,dx - 2\int \sec x\,dx + \int \sec^2 x\,dx - \int 1\,dx \\ &= -\cot x - 2\ln\left|\sec x + \tan x\right| + \tan x - x + C\end{aligned}$$

More examples for you to complete:

1. $\int (\sec x + \cot x)^2\,dx, (\sec x + \cot x)^2 = \sec^2 x + 2\csc x + \csc^2 x - 1$

Answer: $\boxed{\tan x - 2\ln\left|\csc x + \cot x\right| - \cot x - x + C}$

2. $\int \csc x \sin 2x\,dx,\ \csc x \sin 2x = 2\cos x$

Answer: $\boxed{2\sin x + C}$

4. Eliminate a Square Root

Example: $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx$

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$2\sin^2 x = 1-\cos 2x$$

$$\int_0^{\pi} \sqrt{1-\cos 2x}\, dx = \sqrt{2}\int_0^{\pi} \sqrt{\sin^2 x}\, dx = \sqrt{2}\int_0^{\pi} |\sin x|\, dx = \sqrt{2}\int_0^{\pi} \sin x\, dx$$
$$= \sqrt{2}\left[-\cos x\right]_0^{\pi} = 2\sqrt{2}$$

More examples for you to complete:

1. $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos 4x}\, dx, 1+\cos 4x = 2\cos^2 2x$

Answer: $\boxed{\sqrt{2}}$

2. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\, dx, 1+\tan^2 x = \sec^2 x$

Answer: $\boxed{\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}$

5. Reduce an Improper Fraction

Example: $\int \frac{2x^3}{x^2-1}dx$

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-1 & \quad 2x^3 \\ & -\left(2x^3-2x\right) \\ \hline & \qquad\quad 2x \end{array}$$

$$\int \frac{2x^3}{x^2-1}dx = \int \left(2x+\frac{2x}{x^2-1}\right)dx = x^2+\ln\left|x^2-1\right|+C$$

More examples for you to complete:

1. $\int \frac{x}{x+1}dx, \frac{x}{x+1}=1-\frac{1}{x+1}$

Answer: $\boxed{x-\ln\left|x+1\right|+C}$

2. $\int \frac{4x^2-7}{2x+3}dx, \frac{4x^2-7}{2x+3}=2x-3+\frac{2}{2x+3}$

Answer: $\boxed{x^2-3x+\ln\left|2x+3\right|+C}$

6. Separate a Fraction

Example: $\int \frac{1+\sin x}{\cos^2 x}\,dx$

$$\frac{1+\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \sec^2 x + \sec x \tan x$$

$$\int \frac{1+\sin x}{\cos^2 x}\,dx = \int \sec^2 x\,dx + \int \sec x \tan x\,dx = \tan x + \sec x + C$$

More examples for you to complete:

1. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx, \frac{1-x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

Answer: $\boxed{\sin^{-1} x + \sqrt{1-x^2} + C}$

2. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx, \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}} + \frac{1}{x}$

Answer: $\boxed{\sqrt{x-1} + \ln|x| + C}$

7. Multiply by a Form of 1.

Example: $\int \frac{1}{1+\cos x}\,dx$

$$\frac{1}{1+\cos x}=\frac{1}{1+\cos x}\cdot\underbrace{\frac{1-\cos x}{1-\cos x}}_{1}=\frac{1-\cos x}{1-\cos^2 x}=\frac{1-\cos x}{\sin^2 x}=\csc^2 x-\csc x\cot x$$

$$\int \frac{1}{1+\cos x}\,dx=\int \csc^2 x\,dx-\int \csc x\cot x\,dx=-\cot x+\csc x+C$$

More examples for you to complete:

1. $\int \frac{1}{1+\sin x}\,dx,\ \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}=\frac{1-\sin x}{\cos^2 x}=\sec^2 x-\sec x\tan x$

Answer: $\boxed{\tan x-\sec x+C}$

2. $\int \frac{1}{\sec x+\tan x}\,dx,\ \frac{1}{\sec x+\tan x}\cdot\frac{\sec x-\tan x}{\sec x-\tan x}=\sec x-\tan x$

Answer: $\boxed{\ln|\sec x+\tan x|-\ln|\sec x|+C}$

8. Use a Non-basic Substitution

Example: $\int \frac{1}{\sqrt{x}\,(1+x)}\,dx$

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2udu = dx$$

$$\int \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int \frac{1}{u\left(1+u^2\right)} \cdot 2u\,du = 2\int \frac{1}{1+u^2}\,du = 2\tan^{-1} u + C$$

$$= 2\tan^{-1}\left(\sqrt{x}\right) + C$$

More examples for you to complete:

1. $\int x^3\sqrt{x^2+1}dx, u = x^2+1, \frac{1}{2}du = xdx, x^2 = u-1$

Answer: $\boxed{\frac{\left(x^2+1\right)}{5} - \frac{\left(x^2+1\right)}{3} + C}$

2. $\int \frac{1}{\left(1+\sqrt{x}\right)^3}dx, u = 1+\sqrt{x}, 2\left(u-1\right)du = dx$

Answer: $\boxed{\frac{-2}{1+\sqrt{x}} + \frac{1}{\left(1+\sqrt{x}\right)^2} + C}$