3.3. Rates of Change and Behaviors of Graphs

Goals:

1. Find the average rate of change of a function given by a table, a formula or a graph
2. Find intervals of increasing / decreasing, local max; local min; absolute max, absolute min from graphs.

Average Rate of Change of a function.

E.g. Cost of gasoline over the years

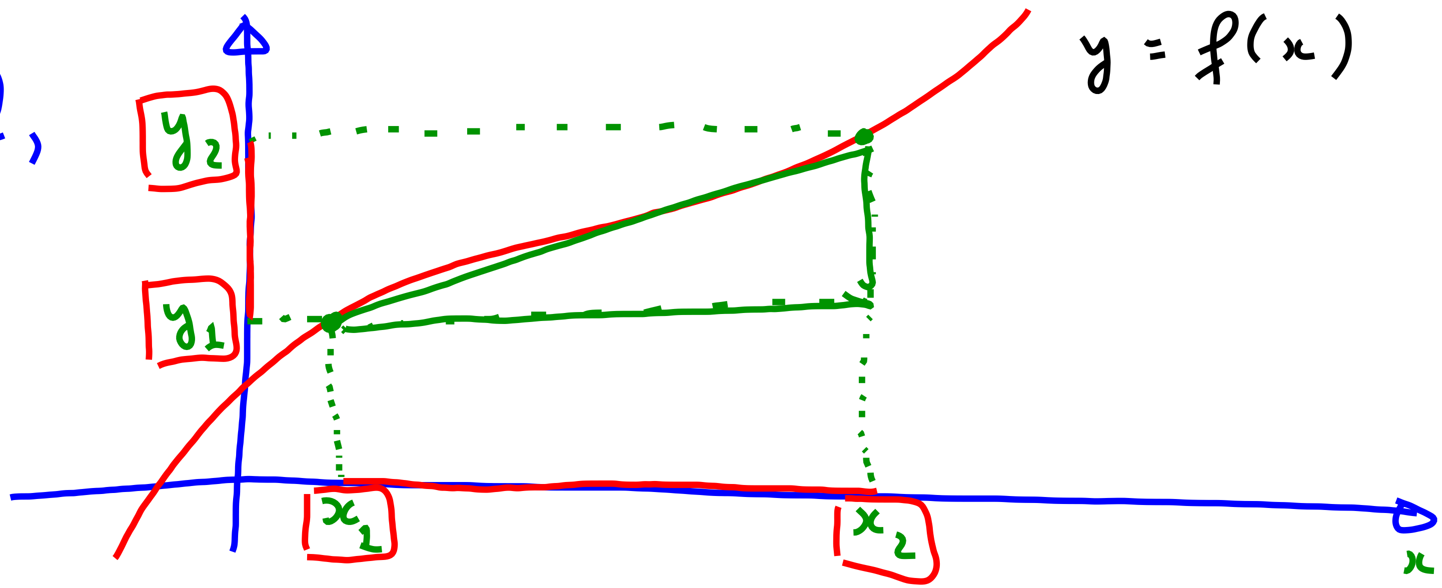
x	2005	2006	2007	2008	2009	2010
$C(x)$	\$2.31	2.62	2.84	3.30	2.41	2.84

Price of gasoline increases by: $\boxed{\$2.84} - \boxed{\$2.31} = \$.53$

Over 5 years, the average increase in gas price:

$$\frac{\boxed{\$.53}}{\boxed{5}} \approx \$.1$$

In general,



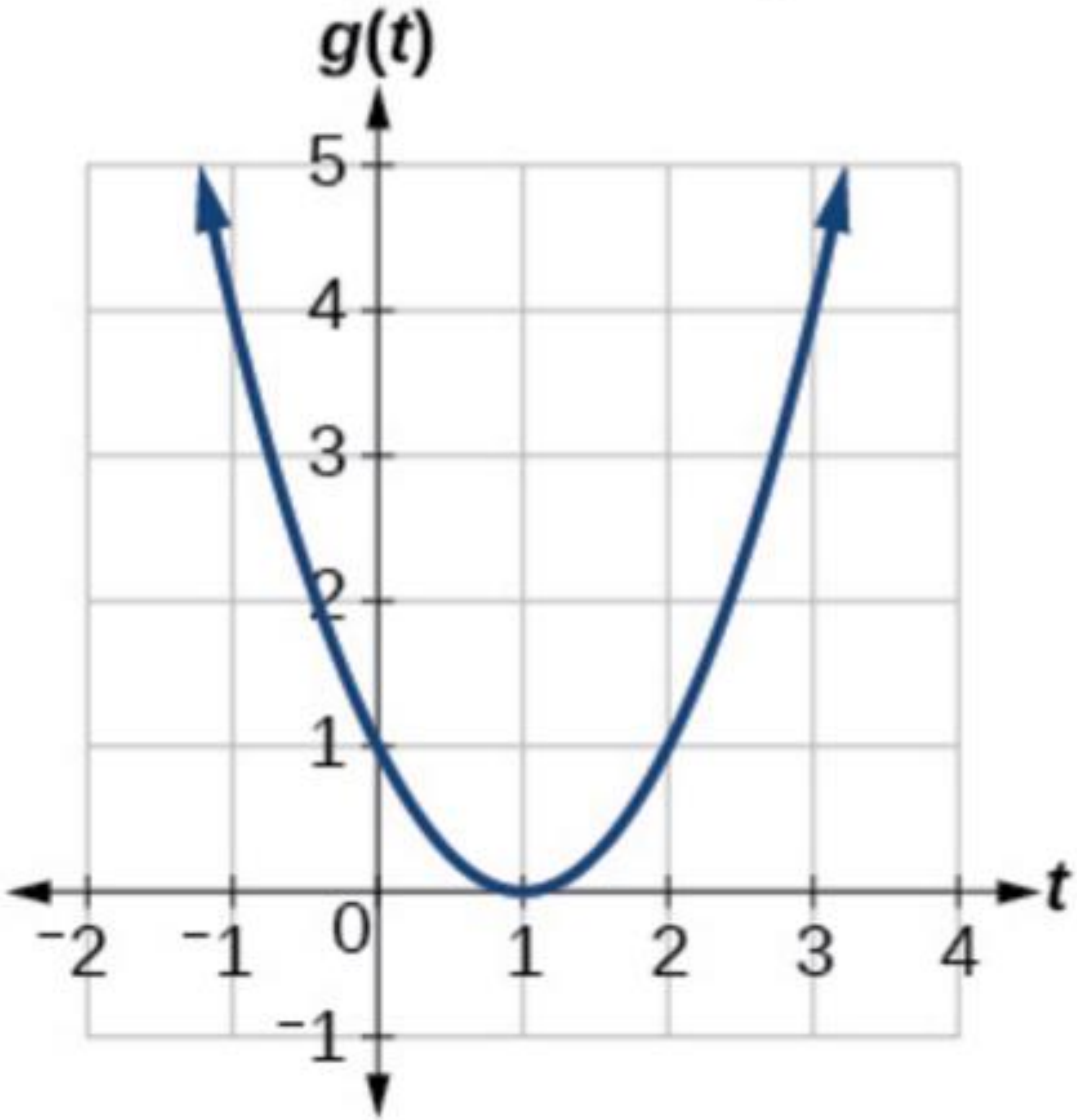
Input	x_1	x_2
Output	y_1	y_2

$y_1 = f(x_1)$; $y_2 = f(x_2)$

Average rate of change of f over $[x_1, x_2]$ $= \dfrac{\text{change in output}}{\text{change in input}} = \dfrac{\Delta y}{\Delta x}$

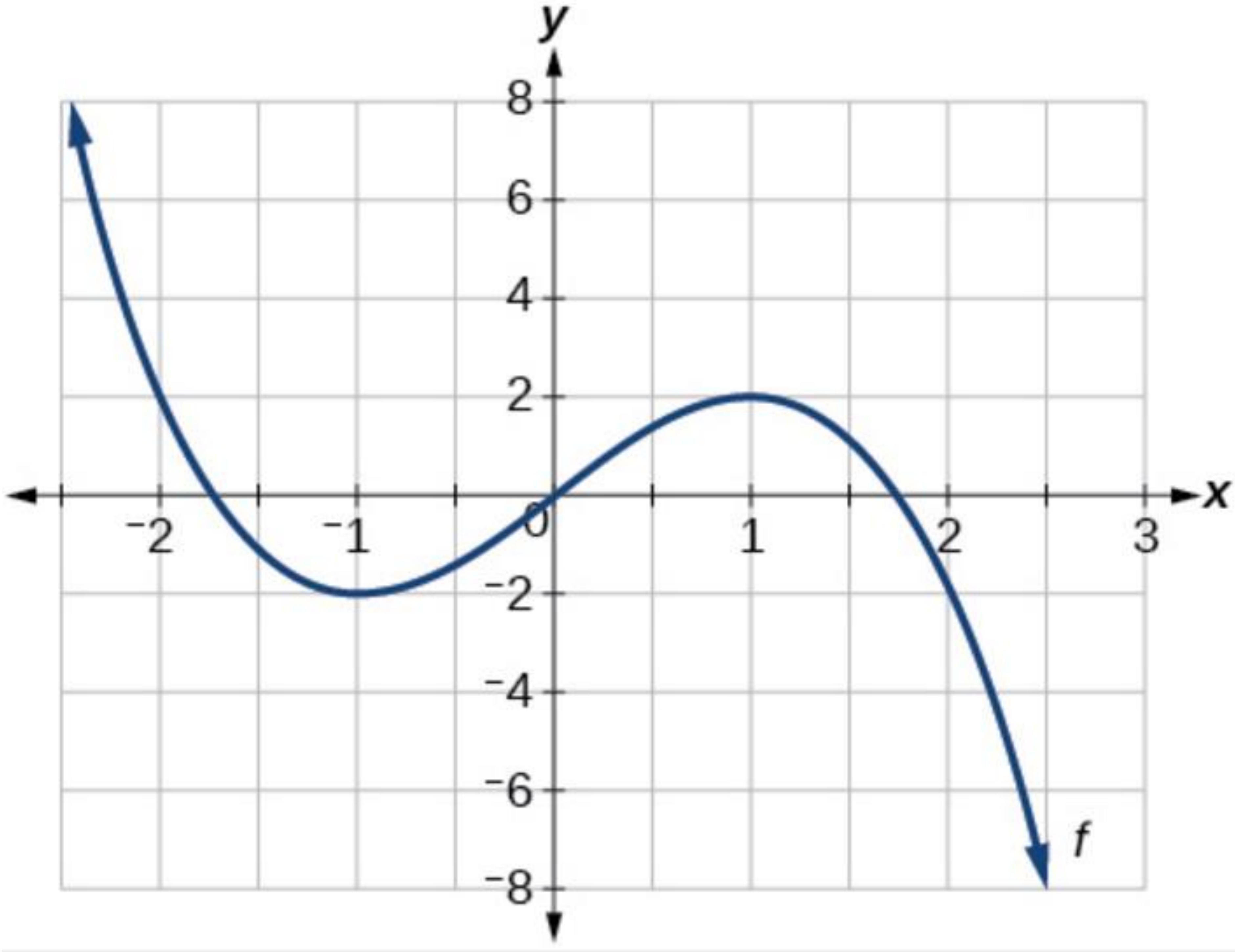
$$= \boxed{\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

Eg.



Average R.O.C over $[-1, 2] = \dfrac{g(2) - g(-1)}{2 - (-1)}$

$= \dfrac{1-4}{3} = \boxed{-1}$



y
x
8
6
4
2
−2
−4
−6
−8
−2
−1
0
1
2
3
f

Average R.O.C. over $[-1, 1] = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{2 - (-2)}{1 + 1}$

$= \dfrac{4}{2} = 2.$

<u>E.g.</u> $h(x) = x^2 - \dfrac{1}{x}.$

Find the average R.O.C. of h over $\boxed{[2, 4]}$

Average R.O.C. on $\boxed{[2, 4]} = \dfrac{h(4) - h(2)}{4 - 2} = \dfrac{\frac{63}{4} - \frac{7 \cdot 2}{2 \cdot 2}}{2}$

$h(2) = (2)^2 - \dfrac{1}{2} = 4 - \dfrac{1}{2} = \dfrac{7}{2}$

$= \dfrac{\frac{63}{4} - \frac{14}{4}}{2}$

$h(4) = (4)^2 - \dfrac{1}{4} = 16 - \dfrac{1}{4} = \dfrac{64}{4} - \dfrac{1}{4} = \dfrac{63}{4}$

$= \dfrac{\frac{49}{4}}{2} = \boxed{\dfrac{49}{8}}$

E.g. $g(x) = x - 2\cdot\sqrt{x}$

Average R.O.C. of g on $\boxed{[1, 9]} = \dfrac{g(9) - g(1)}{9 - 1}$

$g(9) = 9 - 2\cdot\sqrt{9} = 3$

$g(1) = 1 - 2\cdot\sqrt{1} = -1$

$= \dfrac{3 - (-1)}{8} = \dfrac{4}{8} = \dfrac{1}{2}.$

The endpoints of the interval doesn't have to be #, it could be mathematical symbols.

E.g. $h(x) = x^2 + 2x - 8$; over $[5, b]$

Average R.O.C. of h on $[5, b] = \dfrac{h(b) - h(5)}{b - 5}$

$h(b) = \boxed{b^2 + 2b - 8}$

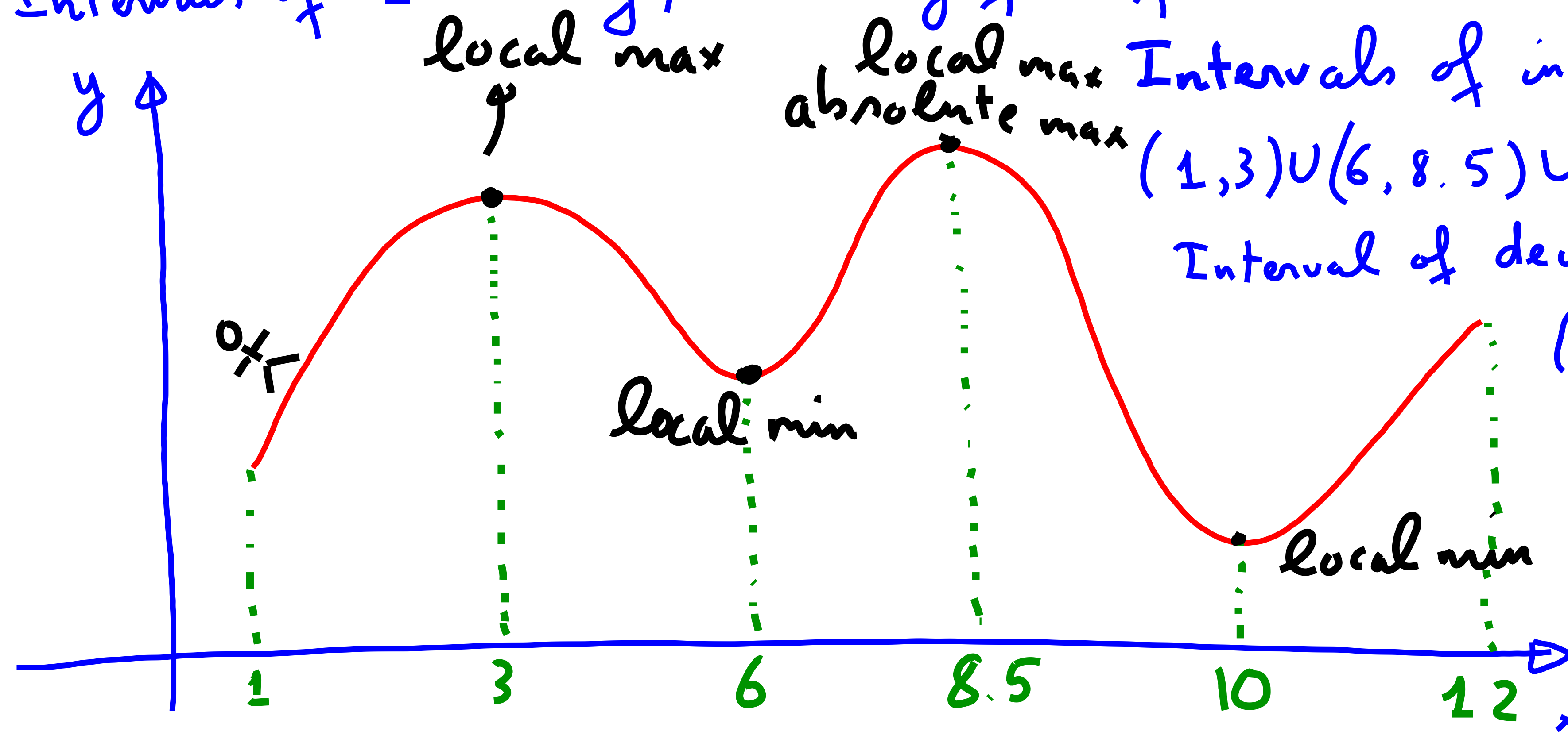
$h(5) = (5)^2 + 2\cdot 5 - 8 = 25 + 10 - 8 = \boxed{27}$

$$\frac{b^2+2b-8-27}{b-5} = \frac{b^2+2b-35}{b-5} = \frac{(b+7)\cancel{(b-5)}}{\cancel{b-5}}$$

$$= b+7$$

Average R.O.C. of h on $[5, b] = b+7$

Intervals of Increasing / Decreasing of a function



Intervals of increasing:

$(1,3) \cup (6, 8.5) \cup (10, 12)$

Interval of decreasing:

$(3, 6) \cup$

$(8.5, 10)$

Eg: $f(x) = 6x^2 + 2$ on $[x, x+h]$

Average R.O.C. $\boxed{\dfrac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}}$

$f(x) = 6x^2 + 2$

$$f(x+h) = 6(x+h)^2 + 2$$
$$= 6(x+h)(x+h) + 2$$
$$= 6(x^2 + xh + xh + h^2) + 2$$
$$= 6(x^2 + 2xh + h^2) + 2.$$

R.O.C. $\dfrac{6(x^2 + 2xh + h^2) + 2 - (6x^2 + 2)}{h}$.

$$\frac{\cancel{6x^2}\ \boxed{+ 12xh + 6h^2}\ \cancel{+2}\ \cancel{-6x^2}\ \cancel{-2}}{h} = \frac{\cancel{h}(12x + 6h)}{\cancel{h}}$$

$\boxed{\begin{array}{c} 6(2x+h) \\ \hline 12x + 6h \end{array}}$

E.g. $a(t) = \dfrac{1}{t+2}$ on $[5, 5+h]$

$$\text{Average R.O.C.} = \frac{a(5+h) - a(5)}{(\cancel{5}+h) - (\cancel{5})}$$

$$a(5+h) = \frac{1}{7+h}$$

$$a(5) = \frac{1}{7}$$

$$\text{R.O.C.} = \frac{\dfrac{1 \cdot 7}{(7+h)\cdot 7} - \dfrac{1 \cdot (7+h)}{7 \cdot (7+h)}}{h}$$

$$= \frac{\dfrac{7}{7(7+h)} - \dfrac{7+h}{7(7+h)}}{h}$$

$$\frac{\dfrac{7-(7+h)}{7(7+h)}}{h} = \frac{\dfrac{\cancel{7}-\cancel{7}-h}{7(7+h)}}{h}$$

$$= \frac{\boxed{\dfrac{-h}{7(7+h)}}}{\dfrac{\boxed{h}}{1}} = \frac{-\cancel{h}}{7(7+h)} \cdot \frac{1}{\cancel{h}} = \boxed{\frac{-1}{7(7+h)}}$$