3.4. Combining and Composing Functions

Goals:
1. **Combining functions:** add, subtract, multiply, divide.
2. **Composing functions:** evaluate composite functions, find domains of composite functions
3. **Decompose functions.**

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1. Combining functions

E.g. $f(x) = x - 1$. $g(x) = x^2 - 1$.

$$(f+g)(x) = f(x) + g(x) = (x-1) + (x^2-1)$$

$$\boxed{(f+g)(x) = x^2 + x - 2}$$

Domain: $(-\infty, \infty)$

$$(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1)$$

$$= x \cancel{-1} - x^2 \cancel{+1}$$

$$(f-g)(x) = -x^2 + x.$$ Domain: $(-\infty, \infty)$

$$(fg)(x) = f(x)\cdot g(x) = (x-1)(x^2-1)$$

$$= x^3 - x - x^2 + 1$$

$$\boxed{(fg)(x) = x^3 - x^2 - x + 1}$$ Domain: $(-\infty, \infty)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-1}{\boxed{x^2-1}}$$

Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$x^2 - 1 = 0$; $x^2 = 1$; $x = \pm 1$

$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \boxed{\frac{1}{x+1}}$$

$x \neq 1$

Ex: $f(x) = \dfrac{1}{x-4}$; $g(x) = \dfrac{1}{6-x}$.

$$(f+g)(x) = f(x) + g(x) = \frac{1\cdot(6-x)}{(x-4)\cdot(6-x)} + \frac{1\cdot(x-4)}{(6-x)\cdot(x-4)}$$

$$= \frac{6-x}{(x-4)(6-x)} + \frac{x-4}{(x-4)(6-x)}$$

Domain: $(-\infty,4)\cup(4,6)\cup(6,\infty)$

$$= \frac{6\cancel{-x}\cancel{+x}-4}{(x-4)(6-x)}$$

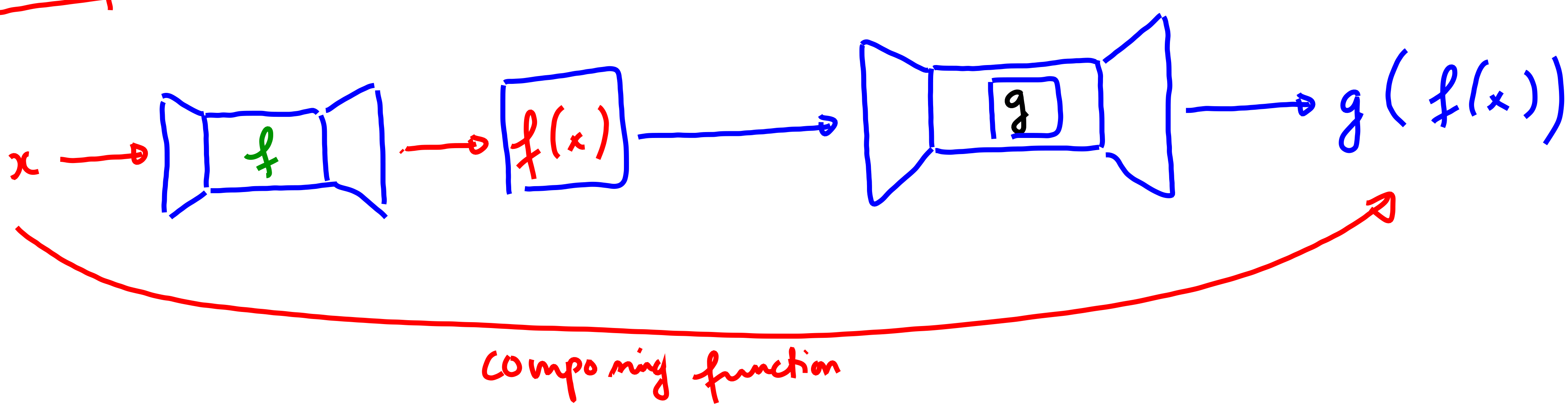
$$(f+g)(x) = \frac{2}{(x-4)(6-x)}$$

$$f(x) = \frac{1}{x-4} \qquad g(x) = \frac{1}{6-x}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x-4}}{\frac{1}{6-x}} = \frac{1}{x-4} \div \frac{1}{6-x}$$

$$= \frac{1}{x-4} \cdot \frac{6-x}{1} = \boxed{\frac{6-x}{x-4}}$$

Composing Functions



f, g

$g \circ f$ (read as g of f or g composed with f)

is a function defined as

$$\boxed{g \circ f(x) = g(f(x))}$$

$g(a) = a+2$

$g(\boxed{a+h}) = a+h+2$

$g(2x) = 2x+2$

E.g. $f(x) = x^2$; $\boxed{g(x) = x+2}$

$g \circ f(x) = g(f(x)) = g(\boxed{x^2}) = \boxed{x^2+2}$

$f \circ g(x) = f(\boxed{g(x)}) = f(\boxed{x+2}) = \boxed{(x+2)^2}$

E.g. $f(x) = x^3 + x^2 + x$

$g(x) = x + 4$

$$\boxed{f \circ g(x)} = f(g(x)) = f(x+4)$$

$$= \boxed{(x+4)^3 + (x+4)^2 + (x+4)}$$

$$\boxed{g \circ f(x)} = g(f(x)) = g(\boxed{x^3 + x^2 + x})$$

$$= \boxed{x^3 + x^2 + x + 4}$$

In general $f \circ g \neq g \circ f$.

E.g.

$$f(x) = \frac{x}{x+2} \; ; \quad g(x) = \frac{x}{x^2-4}$$

$$f \circ g(x) = f(g(x)) = f\left(\boxed{\frac{x}{x^2-4}}\right)$$

$$= \frac{\boxed{\frac{x}{x^2-4}}}{\boxed{\boxed{\frac{x}{x^2-4}} + \frac{2}{1} \cdot \frac{(x^2-4)}{(x^2-4)}}}$$

$$= \frac{\frac{x}{x^2-4}}{\frac{x}{x^2-4} + \frac{2(x^2-4)}{x^2-4}}$$

$$= \frac{\boxed{\frac{x}{x^2-4}}}{\boxed{\frac{x+2x^2-8}{x^2-4}}}$$

$$\frac{x}{\cancel{x^2-4}} \cdot \frac{\cancel{x^2-4}}{2x^2+x-8}$$

$$= \boxed{\frac{x}{2x^2+x-8}}$$

Evaluate composite functions from tables, graphs, formulas.

Did Ex 23, 20, 10

(Table) (Graph) (formula)

Domain of Composite functions

Eg. Find the domain of $f \circ g$ where $f(x) = \dfrac{5}{x-1}$

and $g(x) = \dfrac{1}{x-2}$.

$$f \circ g(x) = f\left(\boxed{g(x)}\right) = f\left(\boxed{\frac{1}{x-2}}\right)$$

$$= \frac{5}{\dfrac{1}{x-2} - \dfrac{1 \cdot (x-2)}{1 \cdot (x-2)}}$$

To find domain :

$x \neq 3$; $x \neq 2$

Domain : $(-\infty, 2) \cup (2,3) \cup (3, \infty)$

$$= \frac{5}{\dfrac{1}{x-2} - \dfrac{x-2}{x-2}} = \frac{5}{\dfrac{1-(x-2)}{x-2}}$$

$$= \frac{5}{\dfrac{1-x+2}{x-2}} = \frac{5}{\boxed{\dfrac{3-x}{x-2}}}$$

$$f \circ g(x) = 5 \cdot \frac{x-2}{3-x} = \boxed{\frac{5x-10}{3-x}}$$

Process to find the domain of $f \circ g(x)$.

1. Find domain of g.
2. Find and simplify $f \circ g(x)$
3. Find domain of $f \circ g(x)$
4. The answer = intersection of the domain in 1 and 3.

Decomposing Functions.

Did Ex 16, 17