5.5. Zeros of polynomials

Goals.

1. Remainder Theorem.
2. Factor Theorem
3. Rational Zero Theorem
4. Fundamental Theorem of Algebra
5. Complex Conjugate Theorem.

Remainder Theorem: Given a polynomial function $f(x)$.

If we divide $f(x)$ by $x-a$, then the remainder is equal to $f(a)$.

If we divide $f(x)$ by $x+a$, then the remainder is equal to $f(-a)$.

E.g. $\underbrace{(6x^4 - x^3 - 15x^2 + 2x - 7)}_{f(x)} \div (x - 1)$

Find the remainder in this division

1st way: Synthetic division.

$$
\begin{array}{c|ccccc}
1 & 6 & -1 & -15 & 2 & -7 \\
  &   & 6 & 5 & -10 & -8 \\
\hline
  & 6 & 5 & -10 & -8 & \boxed{-15}
\end{array}
$$

Remainder $= -15$

2nd way: Remainder Theorem says that

$$\text{Remainder} = f(1)$$

$$f(1) = 6(1)^4 - (1)^3 - 15(1)^2 + 2(1) - 7$$

$$= 6 - 1 - 15 + 2 - 7 = \boxed{-15}$$

E.g. $(3x^5 - 4x^4 - 8x^3 + x^2 - 20x + 3) \div (x+1)$

Remainder Theorem says that

$$\text{Remainder} = f(-1) = 3\cdot(-1)^5 - 4(-1)^4 - 8(-1)^3 + (-1)^2 - 20(-1) + 3$$

$$= -3 - 4 + 8 + 1 + 20 + 3$$

$$\text{Remainder} = 25$$

Why is the remainder Theorem True?

$f(x) \div (x-a)$

Call the quotient $q(x)$ and the remainder r

$$f(x) = (x-a)\cdot q(x) + r$$

Replace x by a : $f(a) = \underbrace{(a-a)}_{0} q(x) + r$

$$f(a) = r$$

Factor Theorem

A number c is a zero of a polynomial $f(x)$ if and only if $(x-c)$ is a factor of $f(x)$.

E.g. $f(x) = x^3 - 6x^2 - x + 30$

Given that $x = -2$ is a zero of $f(x)$.

Find the remaining zeros of f.

Since $x = -2$ is a zero of f; $(x+2)$ must be a factor of f.

To find remaining factor, we need to divide $f(x)$ by $x+2$

$$\begin{array}{r|rrrr} & 1 & -6 & -1 & 30 \\ -2 & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & \boxed{0} \end{array}$$

$x^2 - 8x + 15$

$f(x) = (x+2)(x^2 - 8x + 15)$

To find zeros:

$$\underbrace{(x+2)}\underbrace{(x^2-8x+15)}=0$$

$$x^2-8x+15=0$$

$$(x-3)(x-5)=0$$

$$x=3;\quad x=5$$

Conclusion: The zeros of f are $x=-2$, $x=3$, $x=5$

(Did #3 in HW)

Rational Zero Theorem

If $f(x)$ is a polynomial with integer coefficients, then any rational zero of f must have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

E.g. $f(x) = 2x^3 + x^2 - 4x + 1$.

① Use the Rational Zero Theorem to list all possibilities for rational zero of this polynomial.

$$\frac{p}{q} = \frac{\pm 1}{\pm 2 \,;\, \pm 1} = \frac{1}{2}, -\frac{1}{2}, 1, -1$$

$f\left(\frac{1}{2}\right) \neq 0$ $\quad f\left(-\frac{1}{2}\right) = 3$ $\quad f(1) = 0$; $f(-1) = 4$

$x = 1$ is a zero.

② Find the remaining zeros

$$\begin{array}{c|cccc} 1 & 2 & 1 & -4 & 1 \\ & & 2 & 3 & -1 \\ \hline & 2 & 3 & -1 & 0 \end{array}$$

Remaining factor of f:

$2x^2 + 3x - 1$

To find the remaining zeros

$$2x^2 + 3x - 1 = 0$$

Quadratic formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 2$, $b = 3$; $c = -1$

$$\frac{-3 \pm \sqrt{9 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{17}}{4}$$

The remaining zeros are $\dfrac{-3 + \sqrt{17}}{4}$ and $\dfrac{-3 - \sqrt{17}}{4}$.

Did #7.

Complex conjugate Theorem

What is the complex conjugate of a complex number?

E.g. $1 + 2i$

The complex conjugate of $1+2i$ is the number

$$1 - 2i$$

E.g. $-11i$, $11i$

Theorem: If $a+bi$ is a zero of a polynomial $f(x)$, then the complex conjugate $a-bi$ is also a zero.

E.g. Find the zeros of $(x-3)(x^2-2x+5)=0$

$x - 3 = 0$ or $x^2 - 2x + 5 = 0$

$x = 3$

$$\frac{2 \pm \sqrt{4 - 4.5}}{2}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

Solution (Zeros) are : $x = 3$; $x = 1 + 2i$, $x = 1 - 2i$