5.6. Rational Functions

Goals:
1. Domain and intercepts of rational functions
2. Vertical, Horizontal and Slant Asymptotes of a rational function
3. Graph rational functions

Rational function: is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials

E.g. $f(x) = \dfrac{x^4 + 2x^3 - x^2 + 4x - 3}{x^3 + 2x^2 - x + 1}$; $f(x) = \dfrac{x^2 + 1}{x^2 - 3x + 2}$

$g(x) = \dfrac{3}{x - 2}$; $h(x) = \dfrac{1}{x}$.

Find the domain of a rational function

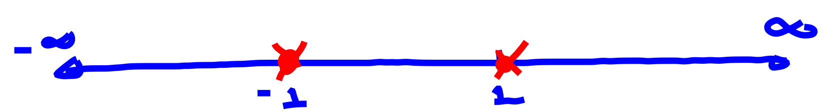
$$f(x) = \frac{p(x)}{q(x)}.$$

* Set $q(x) = 0$. Solve for x.
* Domain: all real numbers except for the value(s) of x you just solved for.

E.g. (1) $f(x) = \dfrac{x-1}{x^2-1}$ (2) $g(x) = \dfrac{x^2+4}{x^2-2x-8}$

(3) $h(x) = \dfrac{x^2+4x-3}{x^4-5x^2+4}$ (4) $u(x) = \dfrac{2x+3}{x^2+16}.$

Find the domain of the given function.



(1) $x^2 - 1 = 0$. $x^2 = 1$. $x = \pm 1$. Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(2) $x^2 - 2x - 8 = 0$. $(x-4)(x+2) = 0$. $x = 4$. $x = -2$ | Domain: $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$

③ $x^4 - 5x^2 + 4 = 0$ $u = x^2$. $u^2 - 5u + 4$

$(u-1)(u-4)$

$(x^2 - 1)(x^2 - 4) = 0$

$x^2 - 1 = 0$ or $x^2 - 4 = 0$

$x^2 = 1$ $x^2 = 4$

$x = \pm 1$ $x = \pm 2$

Domain: $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, \infty)$

④ $x^2 + 16 = 0$ Domain $(-\infty, \infty)$

$x^2 = -16$

$x = \pm 4i$

Find intercepts of a rational function $f(x) = \dfrac{p(x)}{q(x)}$.

Find x-intercept(s):	Set $f(x) = 0$ and solve for x
Set numerator $= 0$	$\dfrac{p(x)}{q(x)} = 0$ implies $p(x) = 0$.

Find y-intercept : Set $x = 0$ in the formula for $f(x)$.

E.g. $f(x) = \dfrac{2x - 3}{3x - 4}$

Find x-intercept(s) and y-intercept of f.

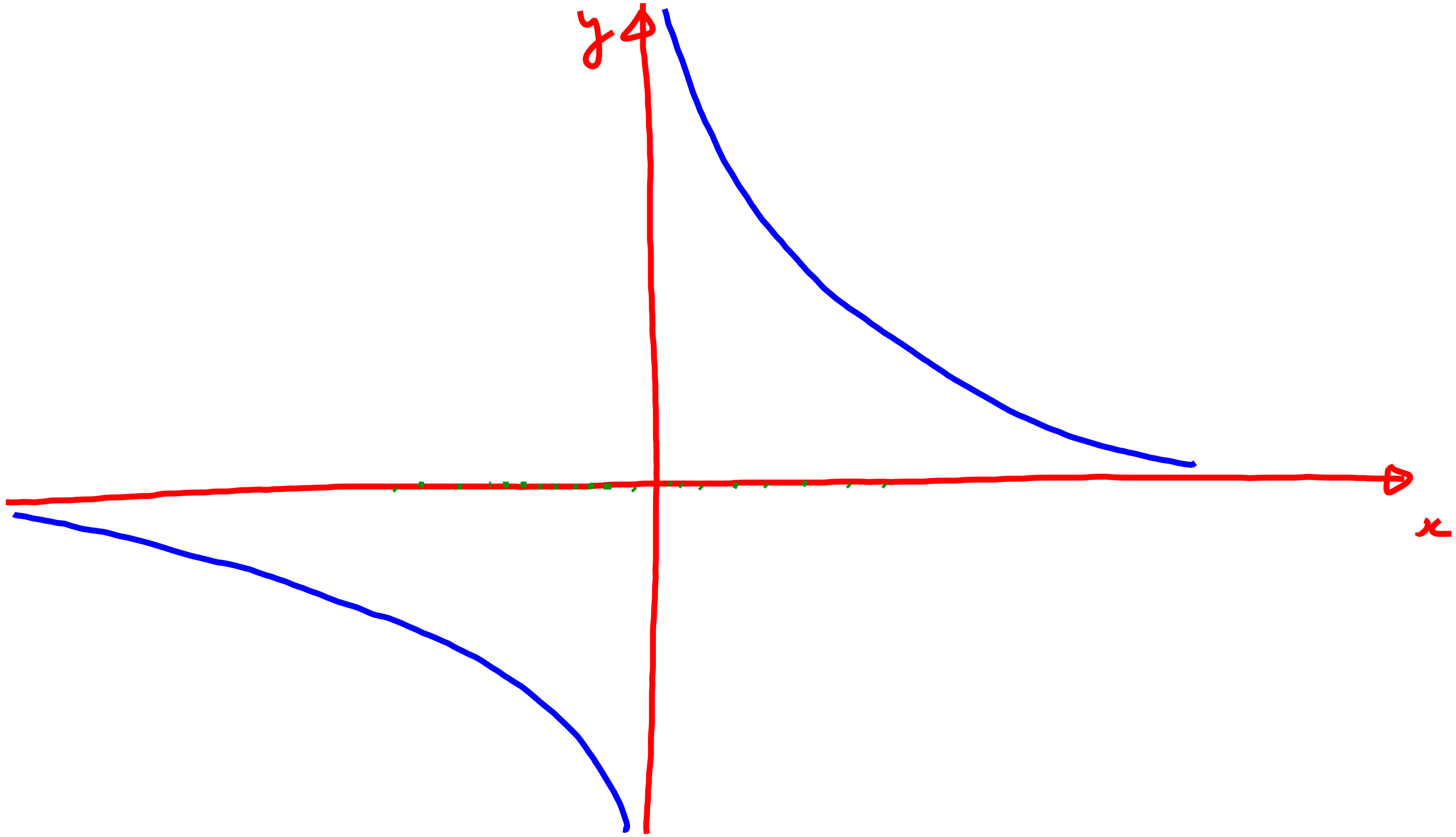
x-intercept : $2x - 3 = 0$; $x = \frac{3}{2}$

$\boxed{\left(\frac{3}{2}, 0\right)}$

y-intercept : Set $x = 0$: $\dfrac{2 \cdot 0 - 3}{3 \cdot 0 - 4} = \dfrac{-3}{-4} = \dfrac{3}{4}$

$\boxed{\left(0, \frac{3}{4}\right)}$

Vertical Asymptotes of Rational functions



$$y = \frac{1}{x}$$

<u>Def of a vertical asymptote</u>: The vertical line $x = a$ is called a vertical asymptote of $y = f(x)$ if as x get close to a, $f(x)$ becomes very very large (positively or negatively)

Process of finding vertical asymptotes of a rational function.

1. Factor the numerator and denominator completely.
2. Cancel the common factor(s) (if any).
3. After cancelation, the values of x which make the denominator of the cancelled function zero correspond to vertical asymptote(s)

The values of x which make the cancelled factor(s) zero correspond to the hole(s) in the graph.

E.g. $$f(x) = \frac{x^2 - 9}{x^2 - 4x + 3}$$

Find the vertical asymptote(s) (V.A.(s)) and holes of this function (if any)

① Factor:

$$f(x) = \frac{(x+3)(x-3)}{(x-3)(x-1)}$$

② Cancel:

$$f(x) = \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x-1)} = \frac{x+3}{x-1}$$

③ $\boxed{x = 1}$ is the V.A.

$x = 3$ corresponds to the hole.

To find y-coord. of the hole, we plug $x = 3$ into the simplified function: $\frac{3+3}{3-1} = \frac{6}{2} = 3$. Hole: $\boxed{(3, 3)}$

Find Horizontal Asymptotes of Rational functions.

<u>Def</u>: The horizontal line $y = a$ is a horizontal asymptote of $y = f(x)$ if $f(x)$ gets very very close to a when x is large.

Process of finding H.A. of $f(x) = \frac{p(x)}{q(x)}$

① Degree of $p(x)$ < Degree of $q(x)$. (E.g. $f(x) = \frac{x^2+1}{3x^5-2}$)

The H.A. is $\boxed{y = 0}$

② Degree of $p(x)$ = Degree of $q(x)$. (E.g. $f(x) = \frac{2x^3 - x^2 + 1}{-5x^3 + x^2 + 2}$)

The H.A. is $y = \frac{a}{b}$ where a is the leading coefficient of $p(x)$

b is the leading coefficient of $q(x)$

(3) Degree $p(x)$ > Degree $q(x)$ (E.g $f(x) = \frac{x^4+1}{x^2+1}$).

No H.A.

To find slant asymptote of a rational function.

$$f(x) = \frac{x^2 - 2x + 5}{x - 1}$$

(Degree of top is exactly one more than degree of bottom)

→ has S.A.

To find the equation of S.A. we use polynomial division.

$$\begin{array}{r|rrr} 1 & 1 & -2 & 5 \\ & & 1 & -1 \\ \hline & 1 & -1 & 4 \end{array}$$

Quotient $= x - 1$ → Slant Asymptote

Remainder $= 4$

Equation of S.A. $y = x - 1$.