6.6. Log and Exponential Equations

Goals: (1) Solve Exp Equations — like bases / Unlike bases

(2) Solve log Equations

Solve Exp Equations with like bases

E.g. $5^{-4x-3} = 5^{-x}$. Solve for x.

Since the bases are the same, the powers must be equal.

$$-4x-3 = -x$$
$$-3x-3 = 0$$

$$-3x = 3$$
$$\boxed{x = -1}$$

Key: If both sides have the same base, to solve for the variable on the exponent, we first equate the exponents, then try to isolate the variable.

E.g.

$$\cancel{9} \cdot 3^{-4n} \cdot \frac{1}{\cancel{9}} = 3^{n+8}\, 9 \qquad \text{Solve for } n.$$

$$3^{-4n} = 3^{n+8} \cdot 9$$

$$3^{-4n} = \underbrace{3^{n+8} \cdot 3^{2}}$$

$$3^{-4n} = 3^{n+10}$$

$$-4n = n + 10$$

$$-5n = 10$$

$$\boxed{n = -2}$$

$$b^{x} \cdot b^{y} = b^{x+y}$$

E.g.

$$256 \cdot \boxed{4^{7x+3}} = 64. \qquad \text{Solve for } x.$$

$$4^{4} \cdot 4^{7x+3} = 4^{3}$$

$$4^{7x+7} = 4^{3}$$

$$7x + 7 = 3$$

$$7x = -4$$

$$\boxed{x = -\frac{4}{7}}$$

2^{nd} way of solving the equation:

$$256 \cdot \boxed{4^{7x+3}} = 64$$

$$4^{7x+3} = \frac{64}{256} = \frac{1}{4}$$

$$4^{7x+3} = \frac{1}{4}$$

$$4^{7x+3} = 4^{-1}$$

$$7x + 3 = -1$$

$$7x = -4$$

$$\boxed{x = -\frac{4}{7}}$$

Solve Exp Equations with unlike bases

E.g. $7 \cdot e^{5t} = 100$. Solve for t.

Divide both sides by 7 to isolate the exponent

$$e^{5t} = \frac{100}{7}$$

(Inverse Property:

$$\log_b\left(b^{\text{Stuff}}\right) = \text{Stuff}$$

$$\ln\left(e^{\text{Stuff}}\right) = \text{Stuff}$$
)

Take ln of both sides

$$\ln\left(e^{5t}\right) = \ln\left(\frac{100}{7}\right)$$

$$5t = \ln\left(\frac{100}{7}\right)$$

$$\boxed{t = \frac{\ln\left(\frac{100}{7}\right)}{5} \approx 0.532}$$

E.g. $6^{x+1} = 3^{6x-1}$. Solve for x.

Take $\ln$ of both sides:

$$\ln\left(6^{x+1}\right) = \ln\left(3^{6x-1}\right)$$

$$(x+1)\cdot\boxed{\ln 6} = (6x-1)\boxed{\ln 3}$$

Power Rule

$\log_b(M^p) = p\log_b(M)$

Distribute:

$$(\ln 6)x + (\ln 6) = (6\ln 3)x - (\ln 3)$$

$$(\ln 6)x - (6\ln 3)x = -\ln 3 - \ln 6$$

$$(\ln 6 - 6\ln 3)x = -\ln 3 - \ln 6$$

$$\boxed{x = \frac{-\ln 3 - \ln 6}{\ln 6 - 6\ln 3} \approx 0.602}$$

E.g Equations Quadratic in form.

$$\boxed{e^{2x}} - e^{x} - 30 = 0$$

$$\left(\boxed{e^{x}}\right)^2 - \boxed{e^{x}} - 30 = 0$$

Let $u = e^x$. The equation becomes

$$u^2 - u - 30 = 0$$

$$(u-6)(u+5) = 0$$

$$u = 6 \quad ; \quad u = -5$$

$e^x = 6$;

$\ln(e^x) = \ln(6)$

$\boxed{x = \ln 6}$

$e^x = -5$

$\ln(e^x) = \ln(-5)$

$\cancel{x = \ln(-5)}$

Cannot take ln of a negative number

The only solution to this equation is $\boxed{x = \ln 6}$

Log Equations

E.g. $7 \cdot \log_4(x) = 14$. Solve for x.

$$\log_4(x) = 2$$

$$\boxed{x = 4^2 = 16}$$

E.g. $2 \cdot \log(6x+4) + 7 = 11$. Solve for x.

$$2\log(6x+4) = 4$$

$$\log(6x+4) = 2$$

$$6x+4 = 100$$

$$6x = 96$$

$$\boxed{x = \frac{96}{6} = 16}$$

$\log(\text{Stuff}) = 2$

$\text{Stuff} = 100$

$$\boxed{\begin{aligned} \log_a(\text{Stuff}) &= b \\ \text{Stuff} &= a^b \end{aligned}}$$

E.g. $\log_3(x^2 - 30) = \log_3(x)$. Solve for x.

$$x^2 - 30 = x$$

$$x^2 - x - 30 = 0$$

$$(x-6)(x+5) = 0$$

$\boxed{x = 6}$; $\cancel{x = -5}$

this is an extraneous sol.
b/c $\log_3(-5)$ doesn't exist.

E.g. $\log(x+2) - \log(x) = \log(51)$. Solve for x.

By quotient rule:

$$\log\left(\frac{x+2}{x}\right) = \log(51)$$

$\frac{x+2}{x} = 51$; $x + 2 = 51x$

$2 = 50x$

$\boxed{x = \frac{2}{50} = \frac{1}{25}}$

E.g. $\ln(x^2 - 13) + \ln(16) = \ln(17)$

$\ln((x^2-13)\cdot 16) = \ln(17)$

$(x^2-13)\cdot 16 = 17$

$16x^2 - 208 = 17$

$16x^2 = 225$

$x^2 = \frac{225}{16}$; $\boxed{x = \pm\frac{15}{4}}$

#24. HW.

$$A = P \cdot e^{rt}$$

$$4500 = 2000 \cdot e^{0.05t}$$

$$e^{0.05t} = \frac{4500}{2000}$$

$$e^{0.05t} = 2.25$$

$$\ln\left(e^{0.05t}\right) = \ln(2.25)$$

$$0.05t = \ln(2.25)$$

$$t = \frac{\ln(2.25)}{0.05} \approx 16.2$$

years.