7.6. Solve System of Equations Using Gaussian Elimination.

(1) System of Linear Equations ⟶ Augmented Matrix

(2) Augmented Matrix ⟶ System of Equations

(3) Gaussian Elimination

E.g.

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

Augmented Matrix $\left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right)$

Coefficient Matrix $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{pmatrix}$

E.g. Given the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array}\right)$$

Write the system of equations with this matrix

$$\begin{cases} x - y + z = 5 \\ 2x - y + 3z = 1 \\ y + z = -9 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightarrow \begin{array}{l} x - y + 2z = 5 \\ y + 3z = 2 \\ z = 1 \end{array}$$

$x - (-1) + 2 \cdot 1 = 5$

$x + 1 + 2 = 5 \;;\; \boxed{x = 2}$

$y + 3 = 2 \;;\; \boxed{y = -1}$

Gaussian Elimination

A way to obtain a very nice augmented matrix from a given one (row echelon form)

$$\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & m & n & q \\ 0 & 1 & p & r \\ 0 & 0 & 1 & s \end{array}\right)$$

E.g.

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right)$$

Use Gaussian Elimination to transform this matrix into row echelon form

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array}\right) \xleftarrow{R_3 + 3R_1}$$

$$\xrightarrow{R_3 + 6R_2} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{array}\right) \xrightarrow{\frac{1}{4}R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{15}{4} \end{array}\right)$$

$$x - 3y + 4z = 3$$

$$y - 2z = 0$$

$$\boxed{z = \frac{15}{4}}$$

$$x - 3\cdot\frac{15}{2} + 4\cdot\frac{15}{4} = 3$$

$$y - 2\,\frac{15}{4} = 0$$

$$y - \frac{15}{2} = 0$$

$$\boxed{y = \frac{15}{2}}$$

$$x = \frac{21}{2}$$