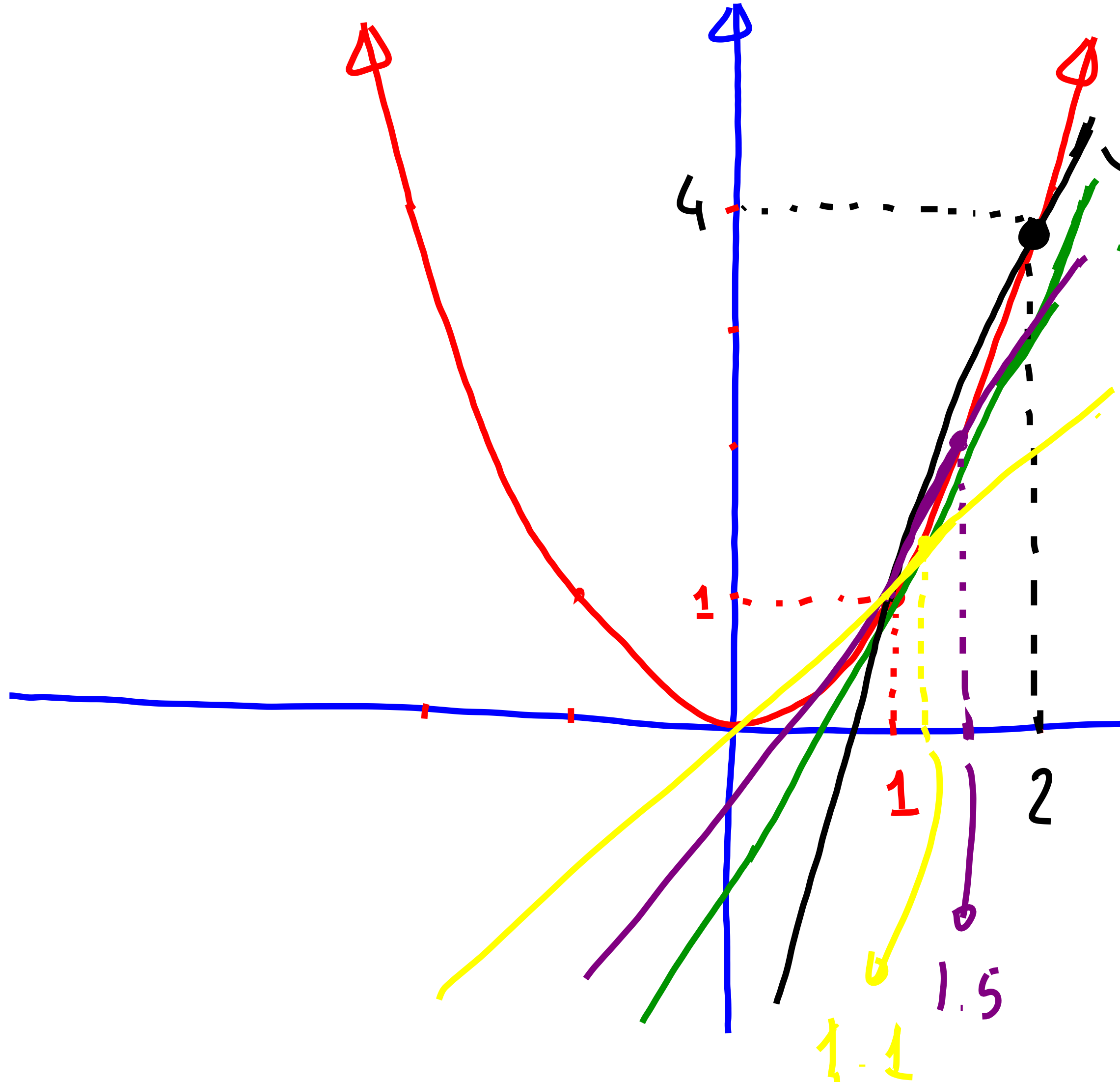
2.1 and 2.2

2.1. A Preview of Calculus

2.2. Limit of a function.

① Tangent line Problem

Problem: $f(x) = x^2$. Find the equation of the tangent line to the graph of $f(x)$ at the point $(1,1)$.



$f(x) = x^2$

secant line

Equation?

Slope?

Slope of secant line through $(1,1)$ and $(2,4)$

$$= \frac{4-1}{2-1} = \frac{3}{1} = 3$$

x	$y = x^2$	m_{sec} (slope of the secant line through $(1,1)$ and (x, y))
2	4	3
1.5	2.25	$\frac{2.25 - 1}{1.5 - 1} = 2.5$
1.25	1.5625	$\frac{1.5625 - 1}{1.25 - 1} = 2.25$
1.1	$(1.1)^2$	$\frac{(1.1)^2 - 1}{1.1 - 1} = 2.1$
1.01	$(1.01)^2$	$\frac{(1.01)^2 - 1}{1.01 - 1} = 2.01$
1.001	$(1.001)^2$	$\frac{(1.001)^2 - 1}{1.001 - 1} = 2.001$

x	y	m_{sec}
a	a^2	$\dfrac{a^2-1}{a-1} = m_{sec}$

(slope of secant line which passes through $(1,1)$ and (a,a^2))

$\boxed{\dfrac{a^2-1}{a-1}} \to g(a)$: function of a.

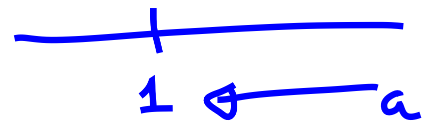
Ask: what does the expression $\boxed{\dfrac{a^2-1}{a-1}}$ get close to as a get closer and closer to 1. ($a = 2;\ 1.5,\ 1.25,\ 1.1,\ 1.01,\ 1.001$)

Mathematical notation to describe this question is

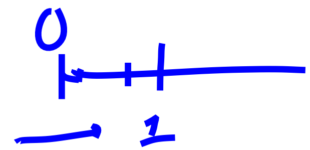
$$\lim_{a \to 1^+} \frac{a^2-1}{a-1} = \lim_{a \to 1^+} g(a)$$

$1 \leftarrow a$

$a \to 1^+$ means a approaches 1 from the right.



We could also ask what happens to $g(a)$ as $a \to 1^-$ (approaches 1 from the left).



$$\lim_{a \to 1^-} \frac{a^2 - 1}{a - 1} = ?$$

$$\lim_{a \to 1^+} \frac{a^2 - 1}{a - 1} = 2$$

a	$g(a) = \frac{a^2 - 1}{a - 1}$
0	1
0.5	1.5
0.6	1.6
0.9	1.9
0.99	1.99
0.999	1.999

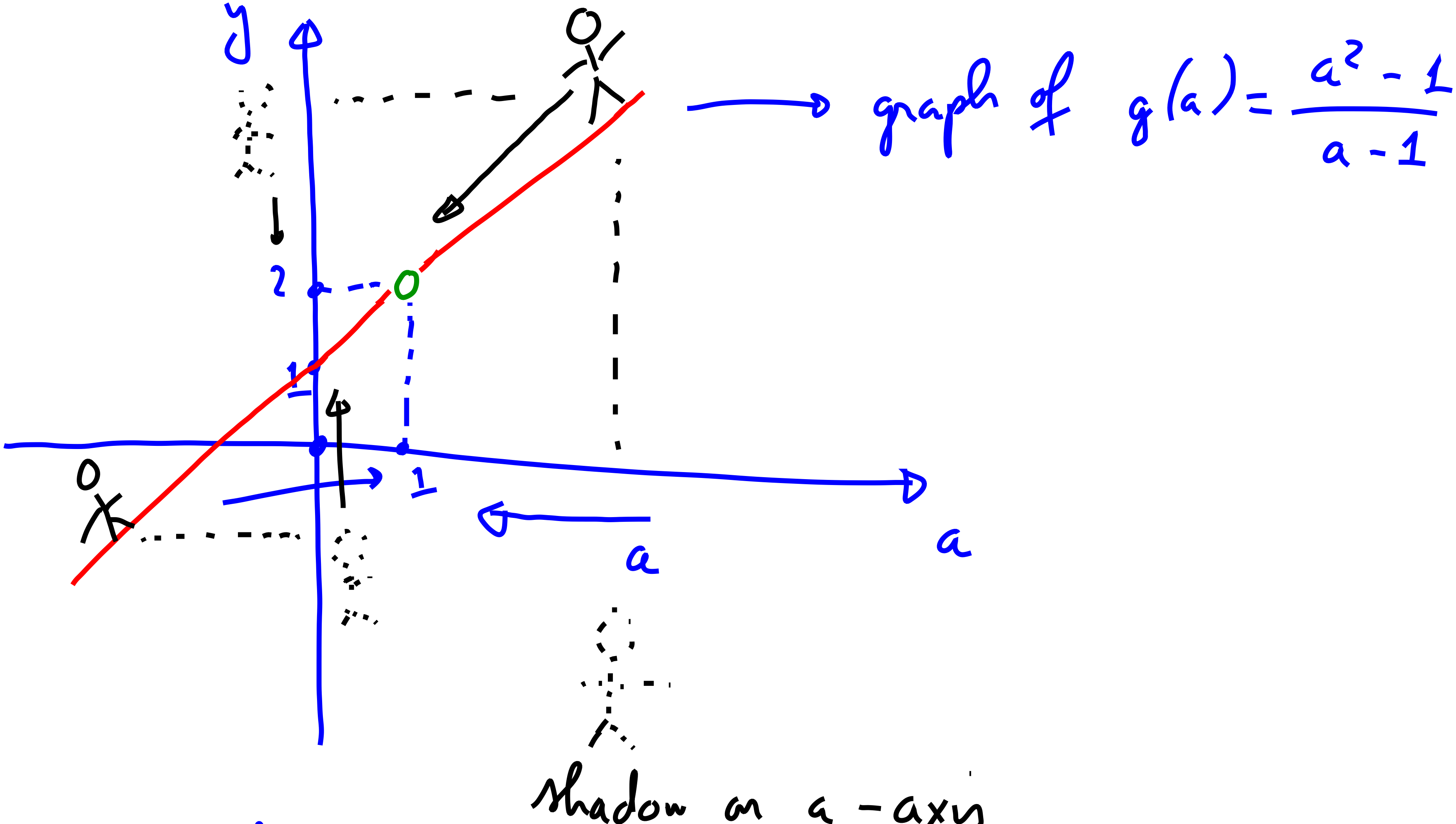
It seems like

$$\lim_{a \to 1^-} \frac{a^2 - 1}{a - 1} = 2$$

* An intuitive explanation of why these limits are equal to 2.

$$\frac{a^2-1}{a-1} = \frac{\cancel{(a-1)}(a+1)}{\cancel{a-1}} = \underbrace{a+1}_{a\neq 1}$$

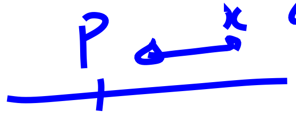
$$\boxed{\frac{a^2-1}{a-1}} = \begin{cases} a+1 & \text{if } a\neq 1 \\ \text{undefined} & \text{if } a=1 \end{cases}$$



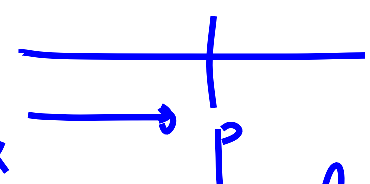
$$\lim_{a \to 1^+} \frac{a^2-1}{a-1} = 2 \qquad \lim_{a \to 1^-} \frac{a^2-1}{a-1} = 2$$

$$\lim_{a \to 1} \frac{a^2-1}{a-1} = 2$$

Definition.

We say that $\lim\limits_{x \to p^+} f(x) = q$ if $f(x)$ gets really really close to q when x gets really really close to p from the right ()

We say that $\lim\limits_{x \to p^-} f(x) = q$ if $f(x)$ gets really really close to q when x gets really really close to p from the left.



We say that $\lim\limits_{x \to p} f(x) = q$ if $\lim\limits_{x \to p^-} f(x) = \lim\limits_{x \to p^+} f(x)$

$= q$.

Back to tangent line problem:

We figured out slope = 2.

Point $(1, 1)$

$$y = mx + b \ ; \quad m = 2$$

$$y = 2x + b$$

Plug in $(1,1)$ to find b: $1 = 2 + b$

$b = -1$

$$y = 2x - 1.$$

Point-slope equation:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$\boxed{y = 2x - 1}$$

Who cares? Physicists.

$s(t) = t^2$: position function of an object.

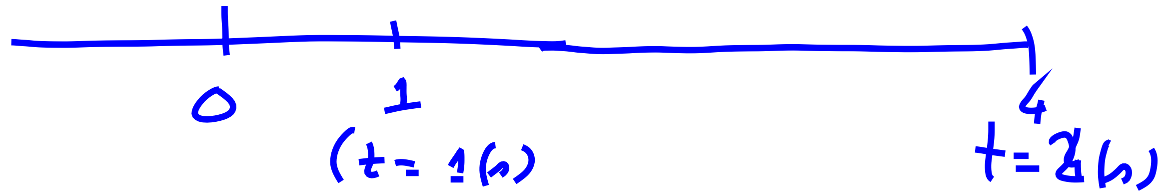
t is measured in seconds. s is measured in m.

$t = 1$; $s(1) = 1$ (m); $t = 2$; $s(2) = 4$ (m)

Average velocity of the object over the time interval $[1, 2]$

$s(1) = 1$

distance



$s(2) = 4$

$$\frac{\boxed{s(2) - s(1)}}{\boxed{2 - 1}} = \frac{4 - 1}{2 - 1} = 3 \ (m/s)$$

time it takes to travel that distance

v_{ave} over $[1, 1.5]$

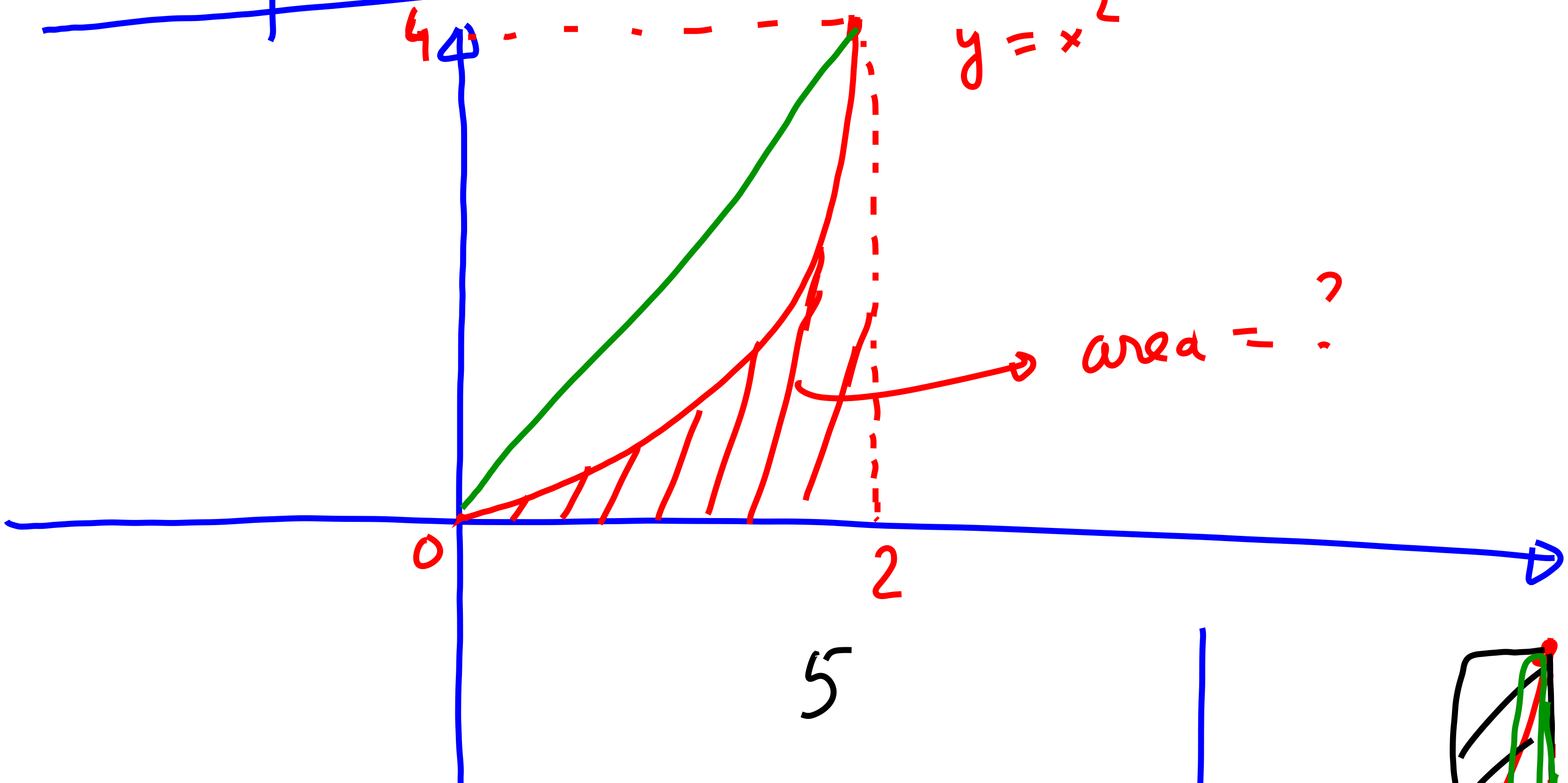
v_{ave} over $[1, 1.1]$

v_{ave} over $[1, 1.01]$

v_{ave} over $[1, 1.001]$

instantaneous velocity at exactly $t = 1 (s)$

② Area problem



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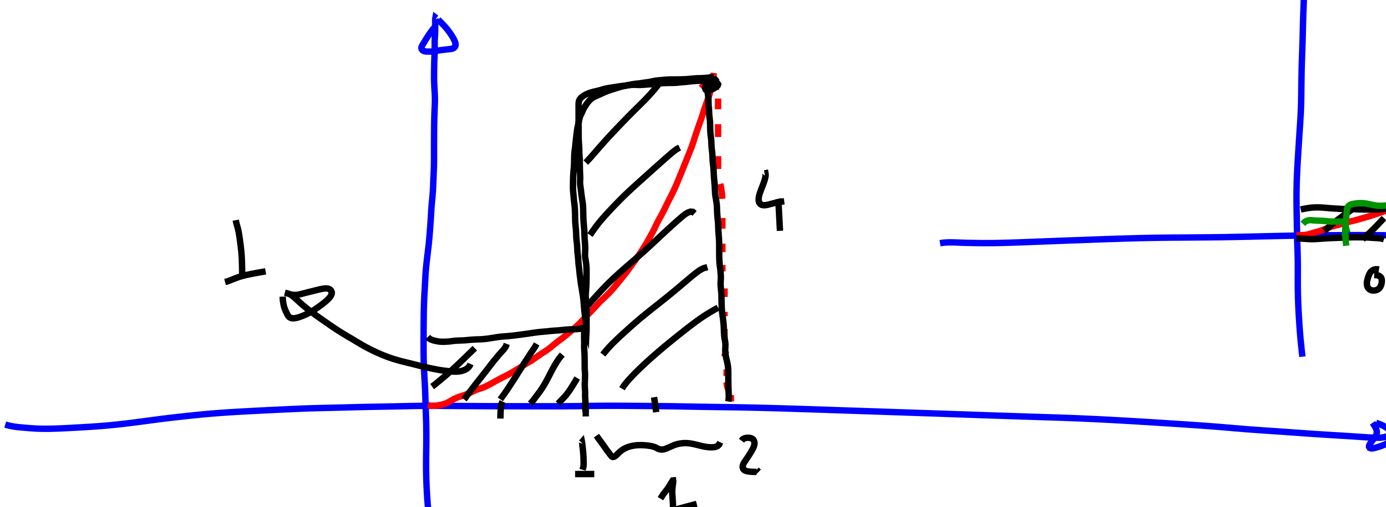
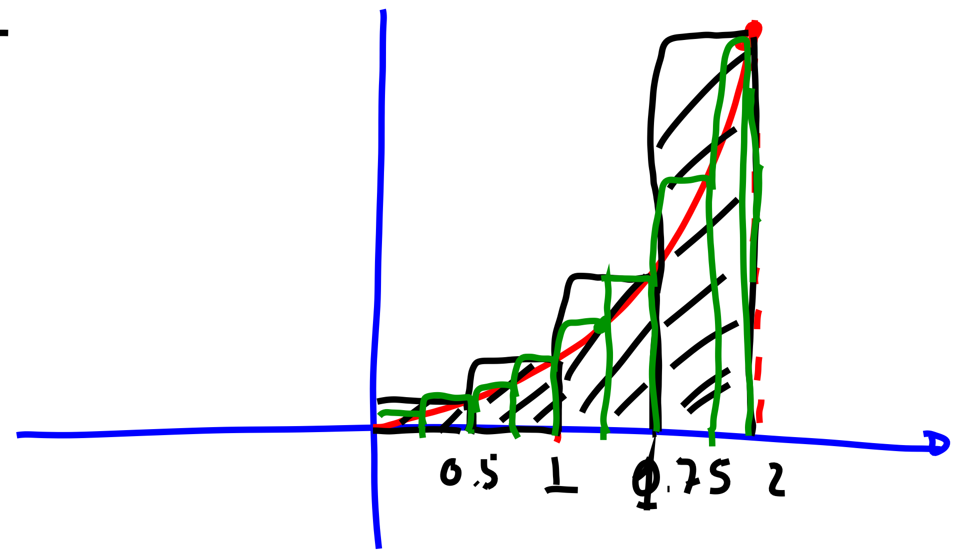




More about limits.

* **Evaluate limits numerically.**

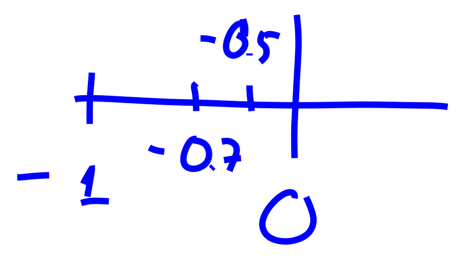
$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

x	$\frac{\sin x}{x}$
1	
0.7	
0.5	
0.1	
0.01	
0.001	

$x \to 0^+$

x	$\frac{\sin x}{x}$
-1	
-0.7	
-0.5	
-0.1	
-0.01	
-0.001	

$x \to 0^-$

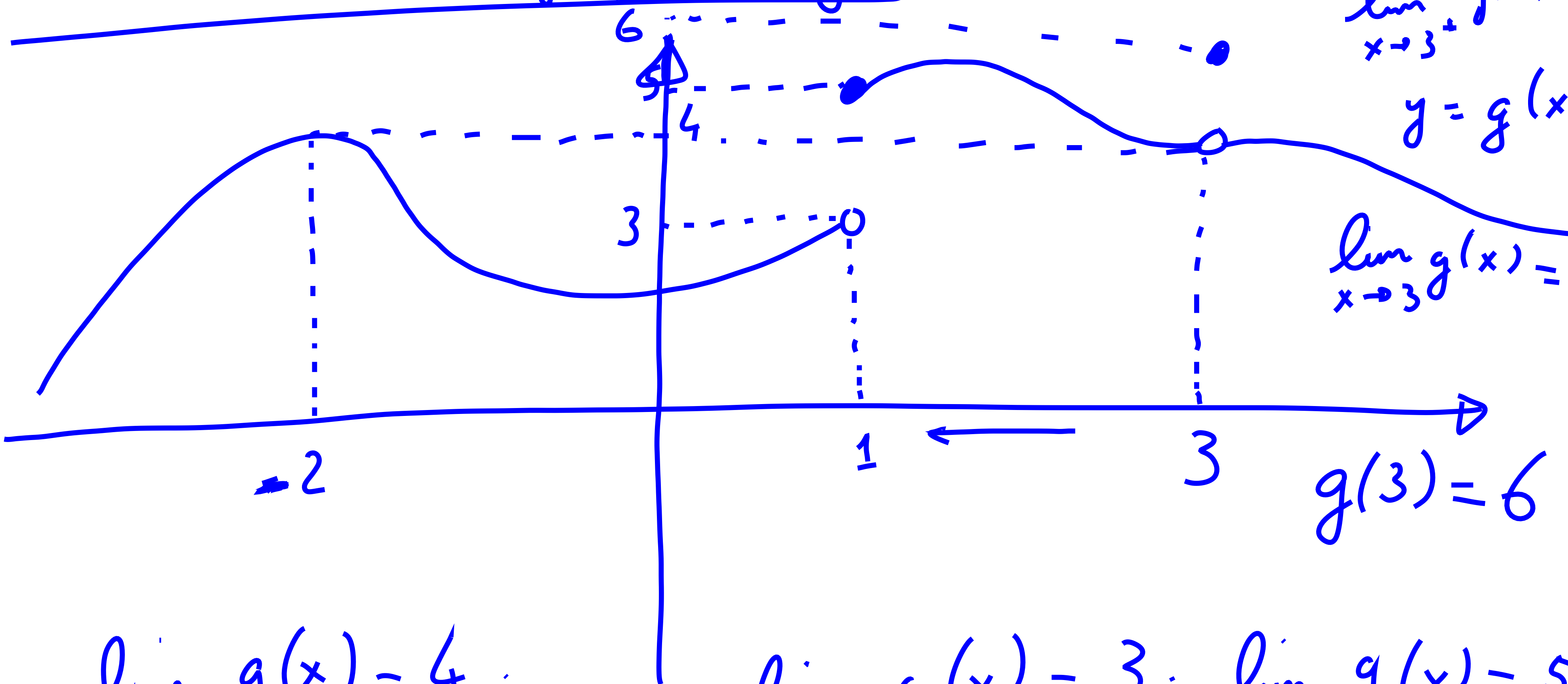


Evaluate limit graphically:

$\lim_{x \to 3^-} g(x) = 4$

$\lim_{x \to 3^+} g(x) = 4$

$y = g(x)$



$\lim_{x \to 3} g(x) = 4$

$g(3) = 6$

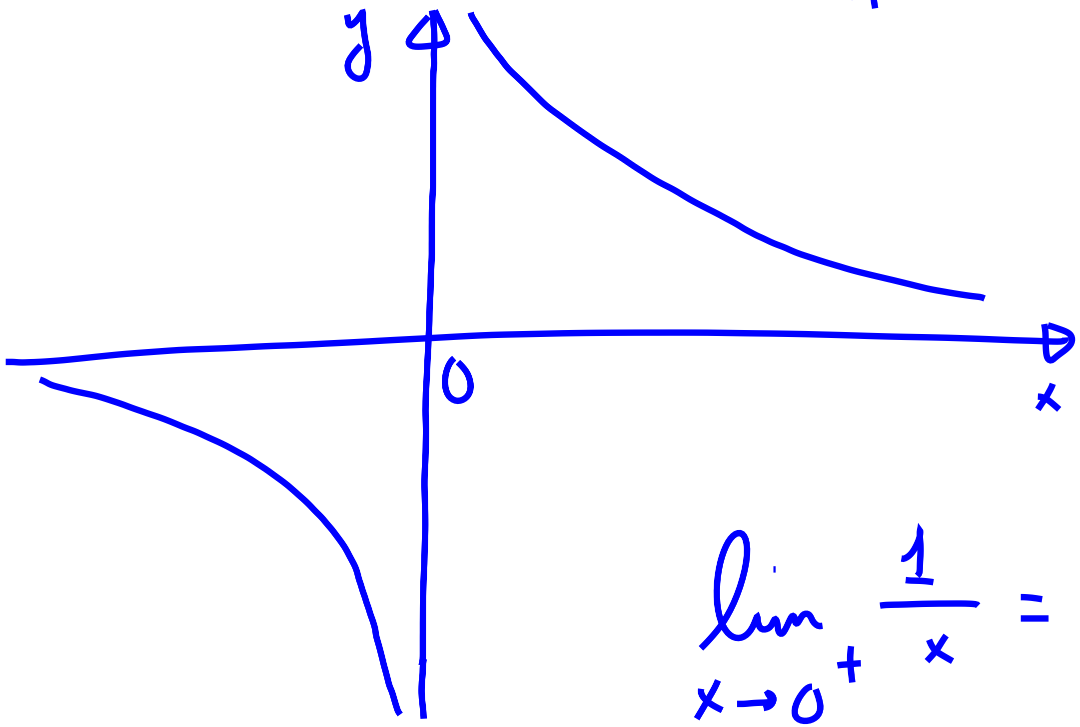
$\lim_{x \to -2} g(x) = 4$;

$\lim_{x \to 1^-} g(x) = 3$; $\lim_{x \to 1^+} g(x) = 5$

$\lim_{x \to 1} g(x)$ DNE

Infinite limits

$$f(x) = \frac{1}{x}$$



x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10000
0.00001	100000

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$