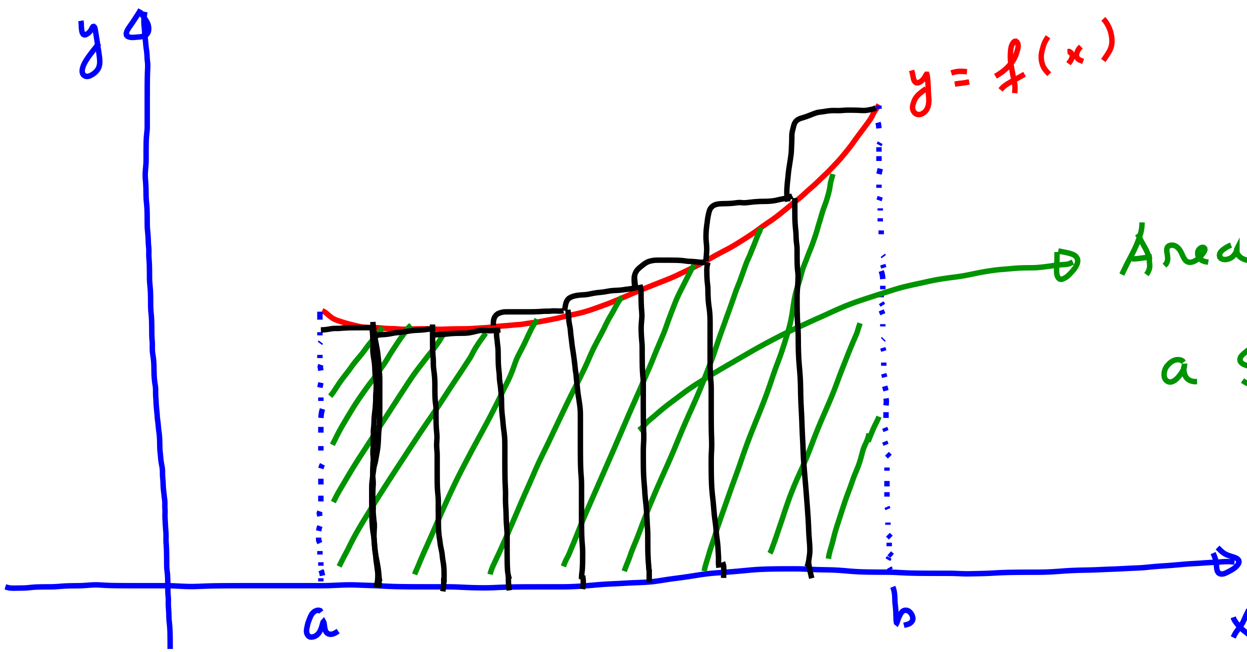
5.1 Areas

Recall: Indefinite integral.

$$\int f(x)\,dx = \text{a family of functions whose derivative equal to } f.$$

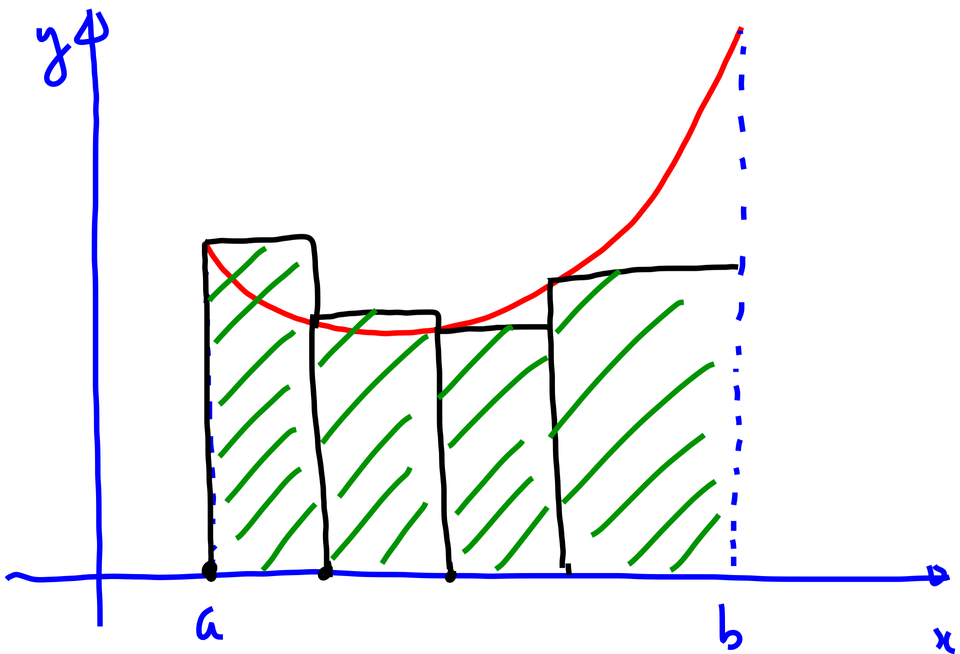
$$= F(x) + C \quad \text{where } F'(x) = f(x).$$

Areas:

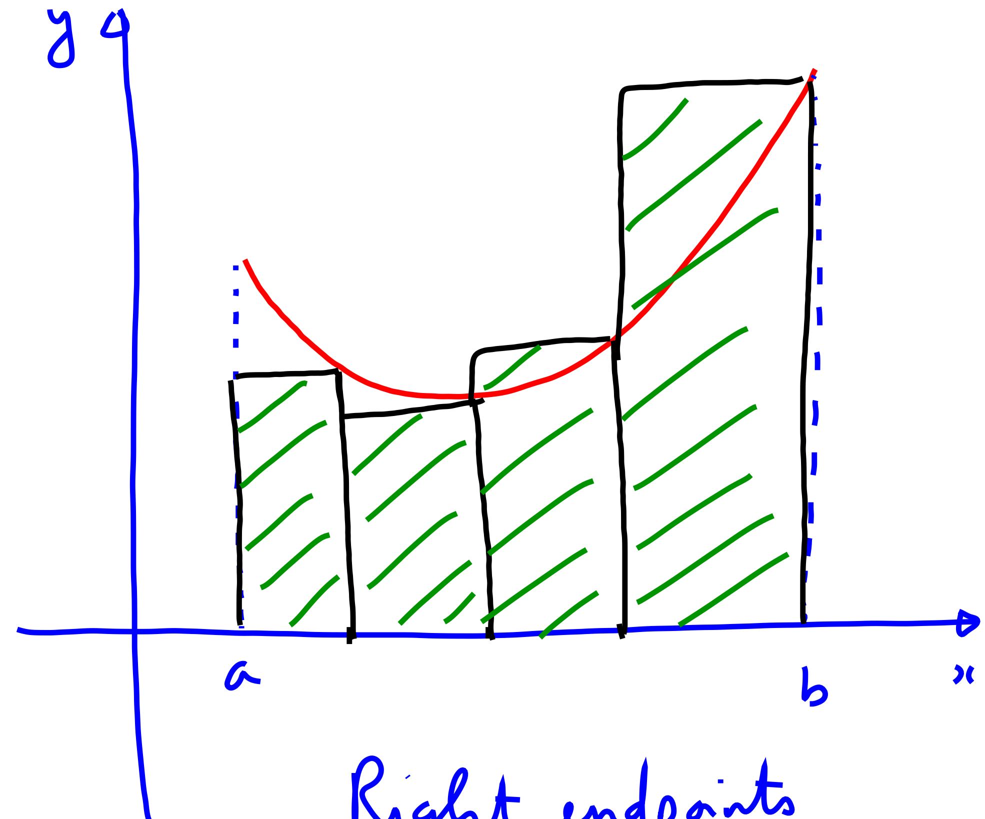


Area under $y = f(x)$; $a \le x \le b$.

* left endpoint and Right Endpoint Approximation of the area under a curve



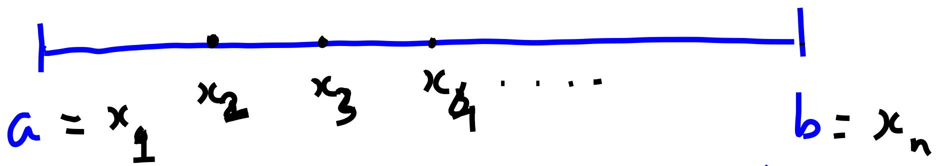
left endpoints



Right endpoints

1. Subdivide $[a, b]$ into n subintervals (of equal length).
   Each subinterval have length $= \Delta x = \dfrac{b-a}{n}$.
2. Draw rectangles from left endpoints (or right endpoints) of these intervals. Each rectangle have width Δx

③ Height of rectangles?

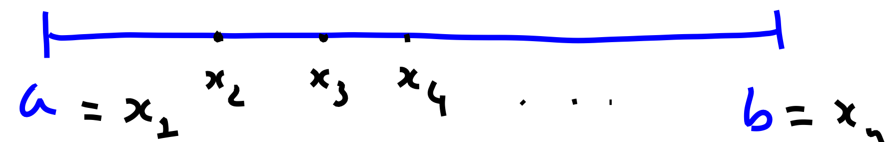


Height of first rectangle : $f(x_1)$

______ second ______ : $f(x_2)$

$\vdots$

Height of last rectangle : $f(x_{n-1})$



Height of first rectangle : $f(x_2)$

$\vdots$

______ last rectangle : $f(x_n)$

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④ Sum of areas of these rectangles :

$$\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)\Delta x$$

$$\left(f(x_2) + \cdots + f(x_n)\right)\Delta x$$

Summation Notation

Left endpoint approximation:

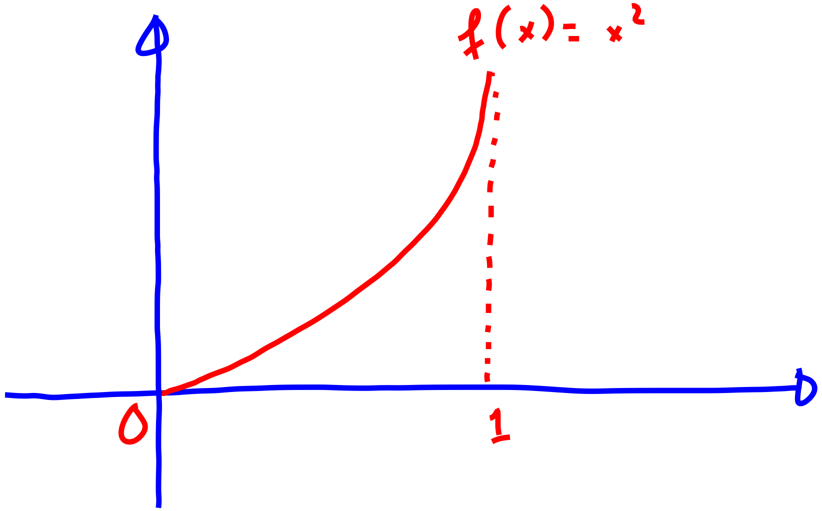
$$L_n = \sum_{i=1}^{n-1} f(x_i)\,\Delta x$$

Right endpoint approximation:

$$R_n = \sum_{i=2}^{n} f(x_i)\,\Delta x.$$

E.g. Consider $f(x) = x^2$. on the interval $[0, 1]$.

Q: Calculate R_4 and L_4

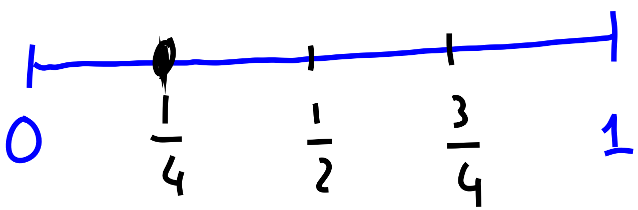


$L_4 =$

$\Delta x = \frac{1}{4}$.

$$\left(f(0) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{4}\right)\right) \cdot \frac{1}{4}$$

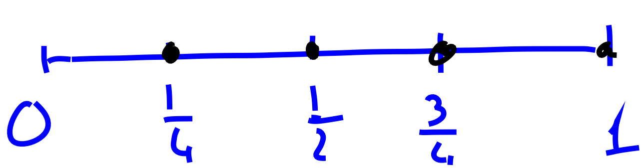
$$\left(0^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{4}\right)^2\right) \cdot \frac{1}{4}$$



$$= \frac{7}{32} \approx 0.21875$$

R_4

$$R_4 = \left(f\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{4}\right) + f(1)\right) \cdot \frac{1}{4}$$



$$= \frac{15}{32} \approx 0.46875$$

So far, $0.21875 < \text{Area} < 0.46875$

* $n = 8$ rectangles:

$$L_8 = 0.2734 < \text{Area} < R_8 = 0.3984$$

* $n = 50$ rectangles

$$L_{50} \simeq 0.3234 < \text{Area} < R_{50} = 0.3434$$

$\longrightarrow$ Area ≈ 0.3

<u>Idea:</u> L_n : depends on n, a function of n

R_n : another function of n.

It has been proved that:

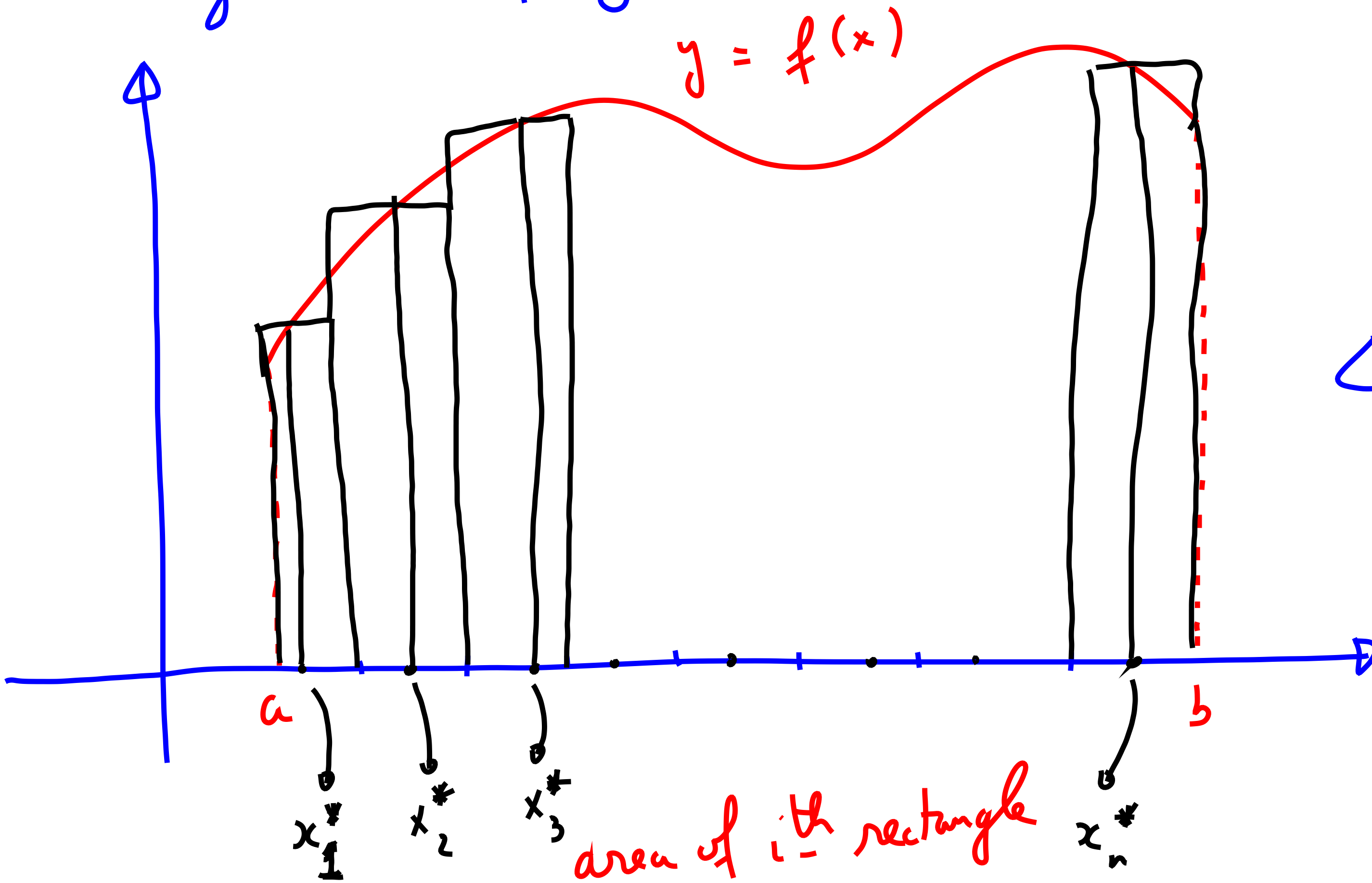
if f is continuous, then:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$ and these limits

are equal to the exact area under $y = f(x)$;

$a \leq x \leq b$.

More general, if $y = f(x)$ is a continuous function on $[a,b]$.



$$\Delta x = \frac{b-a}{n}.$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \overbrace{\underbrace{f(x_i^*)}_{\substack{\text{height of} \\ \text{the } i^{\text{th}} \text{ rectangle}}} \cdot \underbrace{\Delta x}_{\text{width}}}^{\text{area of } i^{\text{th}} \text{ rectangle}}$$

$\longrightarrow$ this limit always exists and it is equal to the area under $f(x)$; $a \leq x \leq b$.

The definite integral of $f(x)$, $a \le x \le b$ is defined to be

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Note: if $f \ge 0$, $\int_a^b f(x)\,dx$ = area under f from a to b.

For general f, $\int_a^b f(x)\,dx$ = signed area.