The Chain Rule

$$\boxed{\frac{d}{dx}\left(x^3\right) = 3x^2}$$

$$\boxed{\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x .$$

$$\boxed{\frac{d}{dx}}\left[\boxed{(\text{Stuff})^3}\right] = \boxed{3(\text{Stuff})^2} \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left(\sqrt{\text{Stuff}}\right) = \frac{1}{2\sqrt{\text{Stuff}}} \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left(\frac{1}{\text{Stuff}}\right) = -\frac{1}{(\text{Stuff})^2} \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left(\cos(\text{Stuff})\right) = -\sin(\text{Stuff}) \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left(\sec(\text{Stuff})\right) = \sec(\text{Stuff}) \cdot \tan(\text{Stuff}) \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left(\tan(\text{Stuff})\right) = \sec^2(\text{Stuff}) \cdot \frac{d}{dx}(\text{Stuff}) .$$

E.g. $$\frac{d}{dx}\left[\left(\underbrace{\boxed{\cos x + x^2 + 7}}_{\text{Stuff}}\right)^3\right] = \boxed{3\left(\underbrace{\boxed{\cos x + x^2 + 7}}_{\text{Stuff}}\right)^2} \cdot \frac{d}{dx}\left(\overbrace{\boxed{\cos x + x^2 + 7}}^{\text{Stuff}}\right)$$

$$= \boxed{3\left(\cos x + x^2 + 7\right)^2 \cdot \left(-\sin x + 2x\right)}$$

E.g. $$\frac{d}{dx}\left(\tan\left(x + \frac{1}{x}\right)\right) = \sec^2\left(x + \frac{1}{x}\right) \cdot \left[1 - \frac{1}{x^2}\right]$$

Chain Rule (2-component chain)

If y is a function of u and u is a function of x

$y \xrightarrow{dy/du} u \xrightarrow{du/dx} x$, then $$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

E.g. $\dfrac{d}{dx}\left(\sqrt{\sin x + \cos x}\right)$

$\boxed{u = \sin x + \cos x}$

$y = \sqrt{u}$

$$y \xrightarrow[dy/du]{} u \xrightarrow[du/dx]{} x$$

$$\frac{d}{du}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}$$

$$\frac{d}{dx}\left(\sin x + \cos x\right) = \cos x - \sin x$$

$$\frac{dy}{dx} = \boxed{\frac{dy}{du}}\cdot\boxed{\frac{du}{dx}} = \frac{1}{2\sqrt{u}}\cdot\left(\cos x - \sin x\right)$$

$$= \boxed{\frac{1}{2\sqrt{\sin x + \cos x}}\cdot\left(\cos x - \sin x\right)}$$

E.g. $\frac{d}{dx}\left[\left(\underbrace{\boxed{x^2+5x+2}}_{u}\right)^{3/2}\right]$

$u = x^2+5x+2$

$\boxed{y = u^{3/2}}$

y — u — x

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{3}{2}u^{1/2}\cdot(2x+5)$$

$$= \boxed{\frac{3}{2}\sqrt{x^2+5x+2}\cdot(2x+5)}$$

Chain Rule with more than 2 components

y — u — v — w — x

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}$$

E.g : $y = \sin(\cos(\sqrt{x}))$; $\dfrac{dy}{dx}$

$u = \sqrt{x}$, $v = \cos(u)$; $y = \sin(v)$

$$y \text{ --- } v \text{ --- } u \text{ --- } x$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \cos(v) \cdot (-\sin(u)) \cdot \frac{1}{2\sqrt{x}}$$

$$= \boxed{\cos(\cos(\sqrt{x})) \cdot (-\sin(\sqrt{x})) \cdot \frac{1}{2\sqrt{x}}}$$

E.g.

① $y = \sin(\sin(\sin x))$. $\dfrac{dy}{dx} = ?$

② $y = \cos\left(\sqrt{\sin(\tan(\pi x))}\right)$. $\dfrac{dy}{dx} = ?$

③ $y = \left[x + (x + \sin^2 x)^3\right]^4$. $\dfrac{dy}{dx} = ?$

In Newton's Notation.

$$y = f(\underbrace{\boxed{g(x)}}_{u})$$

$$y' \overset{?}{=} f'(g(x)) \cdot g'(x)$$

$u = g(x)$

$y = f(u)$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$y' = f'(u) \cdot g'(x)$

$y' = f'(g(x)) \cdot g'(x)$

Mix derivatives.

$$y = \left(\frac{x^2+1}{x^2-1} \right)^3, \quad \frac{dy}{dx} = 3 \cdot \left(\frac{x^2+1}{x^2-1} \right)^2 \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$= 3 \cdot \left(\frac{x^2+1}{x^2-1} \right)^2 \cdot \frac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{(x^2-1)^2}$$

$$= 3 \left(\frac{x^2+1}{x^2-1} \right)^2 \cdot \left(\frac{-4x}{(x^2-1)^2} \right)$$

Eg.

$$G(x) = f\left(x f\left(x f(x)\right)\right)$$

$f(1)=2$; $f(2)=3$; $f'(1)=4$; $f'(2)=5$

$f'(3)=6$.

Question. Find $G'(1)$?