

The Chain Rule

$$\frac{d}{dx} (x^3) = 3x^2$$

$$\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} \left[(\text{Stuff})^3 \right] = 3(\text{Stuff})^2 \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} (\sqrt{\text{Stuff}}) = \frac{1}{2\sqrt{\text{Stuff}}} \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} \left(\frac{1}{\text{Stuff}} \right) = -\frac{1}{(\text{Stuff})^2} \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} (\cos(\text{Stuff})) = -\sin(\text{Stuff}) \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} (\sec(\text{Stuff})) = \sec(\text{Stuff}) \cdot \tan(\text{Stuff}) \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} (\tan(\text{Stuff})) = \sec^2(\text{Stuff}) \cdot \frac{d}{dx} (\text{Stuff})$$

E.g. $\frac{d}{dx} \left[\underbrace{(\cos x + x^2 + 7)}_{\text{Stuff}} \right]^3 = \underbrace{3 \left(\underbrace{(\cos x + x^2 + 7)}_{\text{Stuff}} \right)^2}_{\text{Stuff}} \cdot \frac{d}{dx} \left(\underbrace{\cos x + x^2 + 7}_{\text{Stuff}} \right)$

$$= 3 (\cos x + x^2 + 7)^2 \cdot (-\sin x + 2x)$$

E.g. $\frac{d}{dx} \left(\tan \left(x + \frac{1}{x} \right) \right) = \sec^2 \left(x + \frac{1}{x} \right) \cdot \left[1 - \frac{1}{x^2} \right]$

Chain Rule (2-component chain)

If y is a function of u and u is a function of x

$y \xrightarrow{\frac{dy}{du}} u \xrightarrow{\frac{du}{dx}} x$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex. $\frac{d}{dx} \left(\sqrt{\sin x + \cos x} \right)$

$u = \sin x + \cos x$

$y = \sqrt{u}$

$y \xrightarrow{\quad} u \xrightarrow{\quad} x$
 $\underbrace{\hspace{1cm}}_{dy/du} \quad \underbrace{\hspace{1cm}}_{du/dx}$

$$\frac{d}{du} (\sqrt{u}) = \frac{1}{2\sqrt{u}}$$

$$\frac{d}{dx} (\sin x + \cos x) = \cos x - \sin x$$

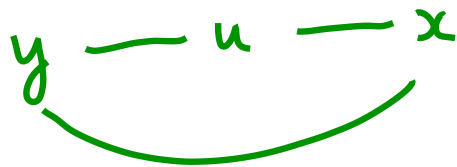
$$\frac{dy}{dx} = \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dx}} = \frac{1}{2\sqrt{u}} \cdot (\cos x - \sin x)$$

$$= \boxed{\frac{1}{2\sqrt{\sin x + \cos x}} \cdot (\cos x - \sin x)}$$

E.g. $\frac{d}{dx} \left[\underbrace{(x^2 + 5x + 2)}_u \right]^{3/2}$

$$u = x^2 + 5x + 2$$

$$y = u^{3/2}$$

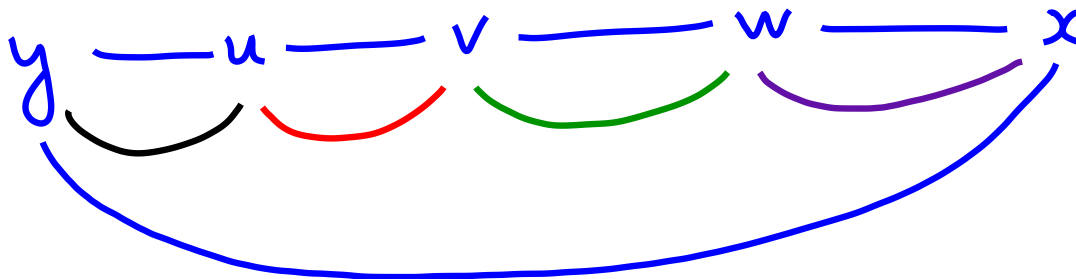


$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{3}{2} u^{1/2} \cdot (2x + 5)$$

$$= \boxed{\frac{3}{2} \sqrt{x^2 + 5x + 2} \cdot (2x + 5)}$$

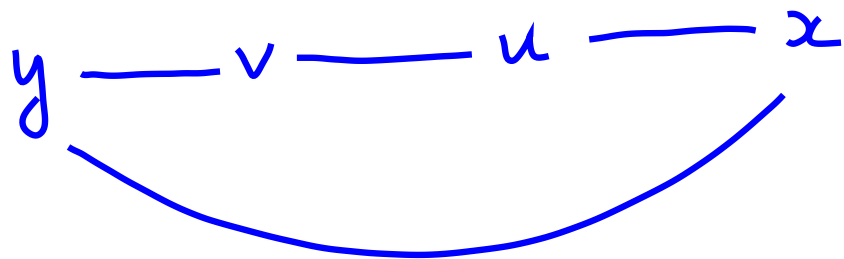
Chain Rule with more than 2 components



$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

E.g : $y = \sin(\cos(\sqrt{x}))$; $\frac{dy}{dx}$

$u = \sqrt{x}$, $v = \cos(u)$; $y = \sin(v)$



$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = \cos(v) \cdot (-\sin(u)) \cdot \frac{1}{2\sqrt{x}}$$

$$= \cos(\cos(\sqrt{x})) \cdot (-\sin(\sqrt{x})) \cdot \frac{1}{2\sqrt{x}}$$

E.g.

$$\textcircled{1} y = \sin(\sin(\sin x)) \quad \frac{dy}{dx} = ?$$

$$\textcircled{2} y = \cos(\sqrt{\sin(\tan(\pi x))}) \quad \frac{dy}{dx} = ?$$

$$\textcircled{3} y = \left[x + (x + \sin^2 x)^3 \right]^4 \quad \frac{dy}{dx} = ?$$

In Newton's Notation.

$$y = f(\overbrace{g(x)}^u)$$

$$y' = f'(g(x)) \cdot g'(x)$$

$$\begin{aligned} u &= g(x) \\ y &= f(u) \end{aligned} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = f'(u) \cdot g'(x)$$

$$y' = f'(g(x)) \cdot g'(x)$$

Mix derivatives.

$$\begin{aligned} y &= \left(\frac{x^2 + 1}{x^2 - 1} \right)^3, \quad \frac{dy}{dx} = 3 \cdot \left(\frac{x^2 + 1}{x^2 - 1} \right)^2 \cdot \frac{d}{dx} \left(\frac{x^2 + 1}{x^2 - 1} \right) \\ &= 3 \cdot \left(\frac{x^2 + 1}{x^2 - 1} \right)^2 \cdot \frac{(x^2 - 1) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 1)^2} \\ &= 3 \left(\frac{x^2 + 1}{x^2 - 1} \right)^2 \cdot \left(\frac{-4x}{(x^2 - 1)^2} \right) \end{aligned}$$

E.g.

$$G(x) = f\left(x f\left(x f(x)\right)\right)$$

$$f(1) = 2; \quad f(2) = 3; \quad f'(1) = 4; \quad f'(2) = 5$$

$$f'(3) = 6.$$

Question. Find $G'(1)$?