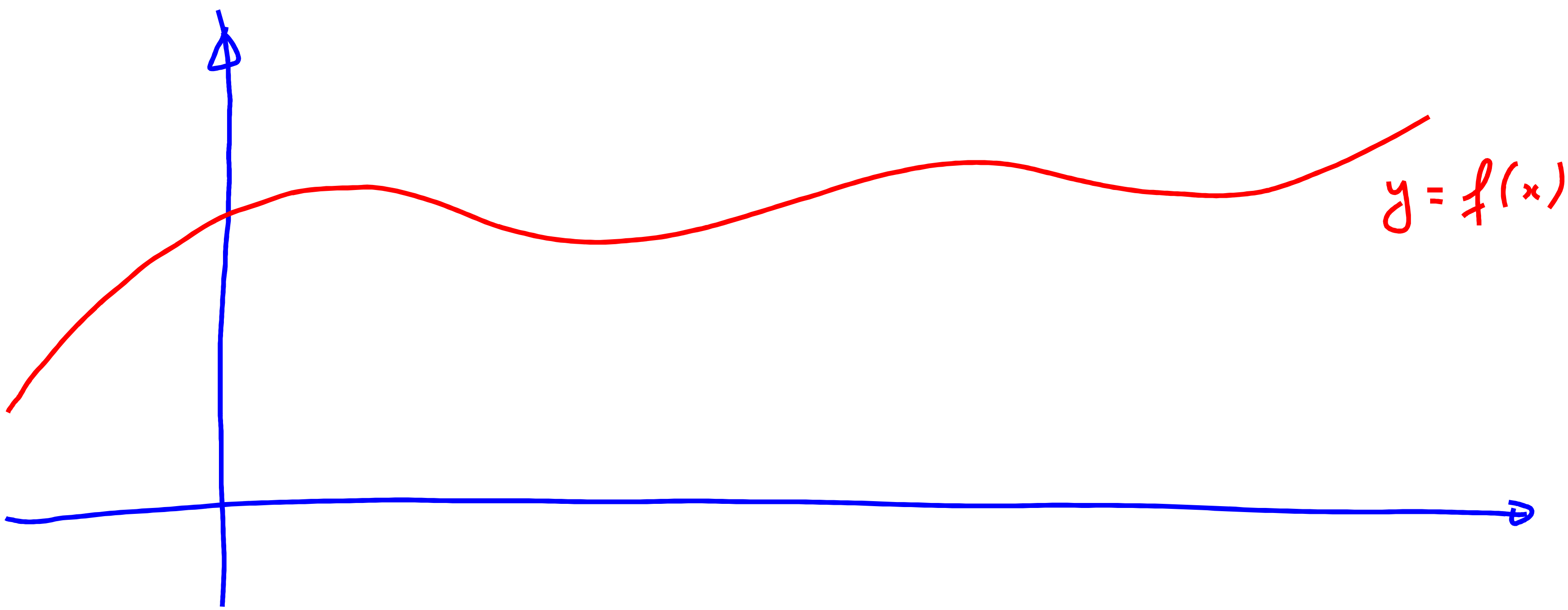
2.4. Continuity

<u>Goal</u>: Understand the concept of continuity rigorously using limits.

Continuity (Intuitive notion)

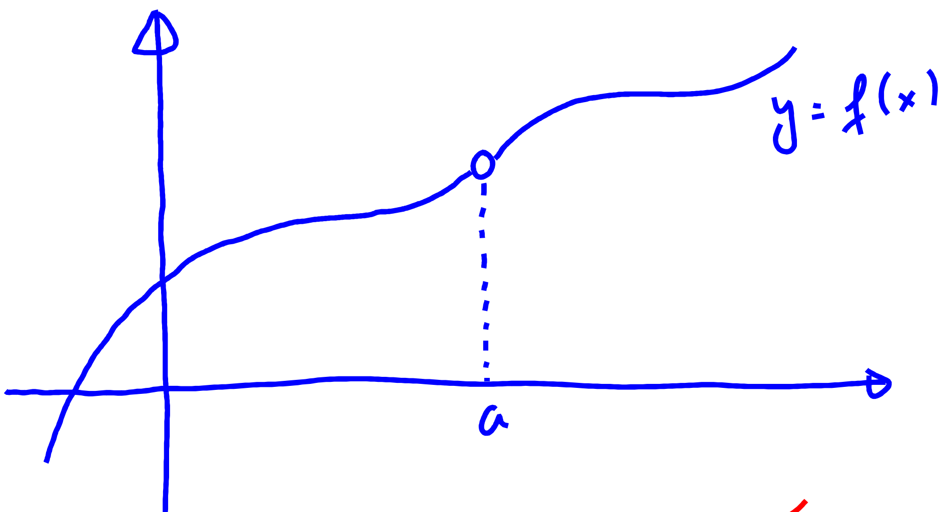


Rigorous notion of continuity

* What does it mean for a function to be continuous at a point?

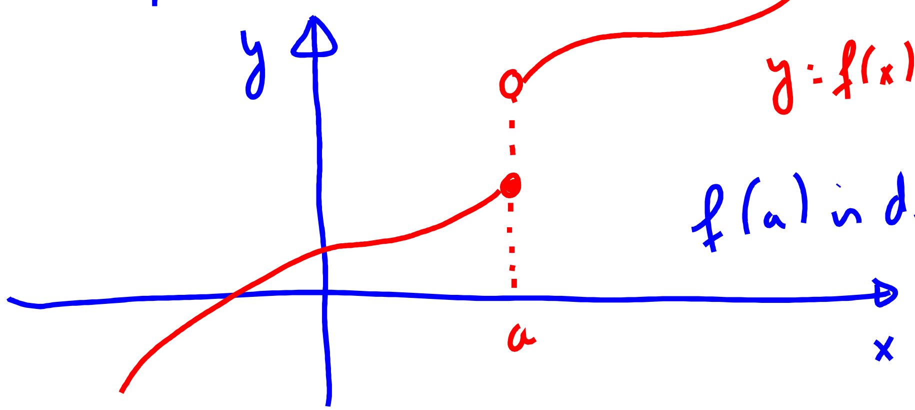
$y = f(x)$. Consider the point $x = a$.

When does f fail to be continuous at $x = a$?



If $f(a)$ is undefined, then f will not be continuous at $x = a$.

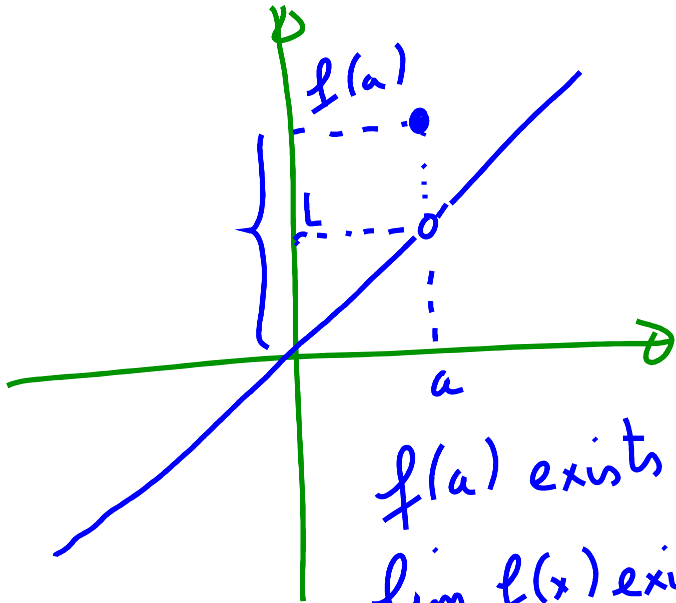
$1^{\underline{st}}$ requirement for continuity: $\boxed{f(a) \text{ is defined.}}$



$f(a)$ is defined

If $\lim\limits_{x \to a} f(x)$ DNE, then f will not be continuous at $x = a$.

$2^{\underline{nd}}$ requirement for continuity: $\boxed{\lim\limits_{x \to a} f(x) \text{ must exist}}$.



$f(a)$ exists

$\lim_{x \to a} f(x)$ exists

but f is still not cont.

3rd requirement for continuity: $\boxed{\lim_{x \to a} f(x) = f(a)}$

<u>Definition</u>: A function $f(x)$ is continuous at a point $x = a$ if and only if the following conditions are satisfied.

1. $f(a)$ must be defined
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$.

E.g. Let $f(x) = \begin{cases} \boxed{x^2 - e^x} & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$

Use the above definition to prove that f is continuous at $x = 0$.

1. Is $f(0)$ defined? $f(0) = -1$ ✓

2. Does $\lim_{x \to 0} f(x)$ exist? $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (x^2 - e^x) = -1$.

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x - 1) = -1$

$\lim_{x \to 0} f(x)$ exist and it is equal to 1 ✓

3. $f(0) = \lim_{x \to 0} f(x)$. Therefore, f is continuous at $x = 0$.

Ex: $g(x) = \dfrac{(x-1)(x+1)}{x-1}$. NOT cont. at 1.   NO. $g(1)$ is undefined

① Is $g(x) = \dfrac{2x^2 - 5x + 3}{x - 1}$ continuous at $x = 1$? Why ?

② Is $k(x) = \begin{cases} 3x & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$ cont. at $x = 1$? Why ?

NO. $\lim\limits_{x \to 1} k(x)$ DNE $\begin{cases} \text{left limit} = 3 \\ \text{right limit} = 1 \end{cases}$

③ Is $h(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ cont. at $x = 0$? Why ?

Sol: $h(0)$ is defined. $h(0) = 1$

$\lim\limits_{x \to 0} h(x) = \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

$\Bigg\}\ \lim\limits_{x \to 0} h(x) = h(0)$

$\Longrightarrow h$ is cont. at $x = 0$.

Def: We say that f is continuous on an interval I

($I = (a,b)$; $I = [a,b]$; $I = [a,b)$; $I = (a,b]$;

$I = (-\infty, \infty)$; $I = (a, \infty)$ . . .)

if f is continuous at every point in I.

Theorem: Polynomial functions and rational functions are always continuous on their domains.

E.g $f(x) = x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1$

Domain: $(-\infty, \infty)$. **Thm**: f is cont. on $(-\infty, \infty)$.

E.g $g(x) = \dfrac{x^2 - 1}{x - 1}$. **Domain**: $(-\infty, 1) \cup (1, \infty)$

Thm: g is cont. on $(-\infty, 1) \cup (1, \infty)$

Prove the Theorem:

Suppose that we have a polynomial function:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 .$$

$a_n, a_{n-1}, \ldots, a_1, a_0$: constants (coefficients)

Take an arbitrary point b in $(-\infty, \infty)$

$$\underline{\lim_{x \to b} p(x)} = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = \underline{p(b)}$$

limit laws: Sum, const. mult., power law.

By the def of cont. at a point, $p(x)$ is cont. at b.

* Rational function: $f(x) = \dfrac{p(x)}{q(x)}$. $\lim_{x \to b} f(x) = \lim_{x \to b} \dfrac{p(x)}{q(x)}$

where p, q are poly.

if $\boxed{q(b) \neq 0}$ $= \dfrac{\lim_{x \to b} p(x)}{\lim_{x \to b} q(x)} = \dfrac{p(b)}{q(b)} = f(b)$

In general, any function that we deal with in this class will be continuous on its domain.

E.g. $f(x) = \sqrt{\boxed{x-5}}$

Domain: $x - 5 \geq 0$

$x \geq 5$

Domain: $[5, \infty)$

f is continuous on $[5, \infty)$

E.g. $g(x) = \tan x$.

Domain: $x \neq \frac{k\pi}{2}$; k: odd integers.

g is cont. at every point except for those points.

$\boxed{\text{Interval of continuity = Domain}}$.

Ex: Find the intervals of continuity of the given function:

① $\boxed{f(x) = \dfrac{5}{e^x - 2}}$

③ $h(x) = \dfrac{2}{x^2 + 5}$.

Interval of cont. = Domain
$= (-\infty, \infty)$

② $g(x) = \sqrt{\boxed{x^3 - x}}$.

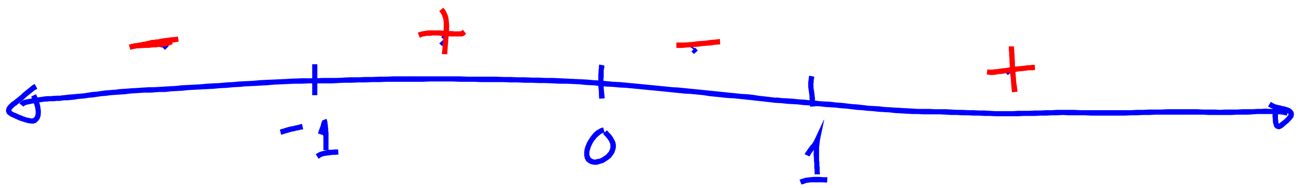
① $e^x - 2 = 0$; $e^x = 2$; $x = \ln(2)$

Domain: $(-\infty, \ln(2)) \cup (\ln(2), \infty)$ ← interval of continuity.

② To find domain: $\boxed{x^3 - x} \geq 0$; $x(x-1)(x+1) \geq 0$

$x(x^2 - 1) \geq 0$

$0 \quad 1 \quad -1$



$[-1, 0] \cup [1, \infty] \leftarrow$ interval of continuity.

Intermediate Value Theorem

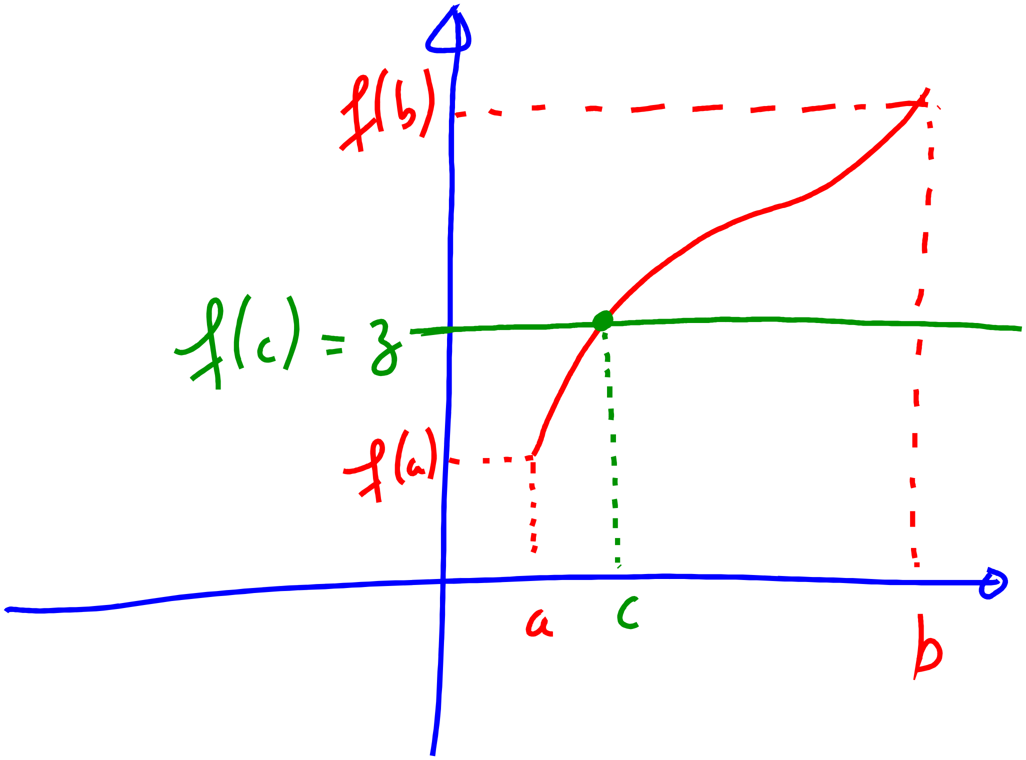
f : continuous on a closed, bounded interval $[a, b]$.

z : any number in between $f(a)$ and $f(b)$; that is,

$$f(a) < z < f(b)$$

Then, there exists a number c in $[a, b]$ such that

$$f(c) = z$$



Special case: if f is cont. on $[a,b]$ and $f(a) < 0$ and $f(b) > 0$, then there exists a # c in $[a,b]$ such that $f(c) = 0$.

E.g. $f(x) = x^3 - x^2 - 3x + 1$.

Show that f has a zero in the interval $[0, 1]$

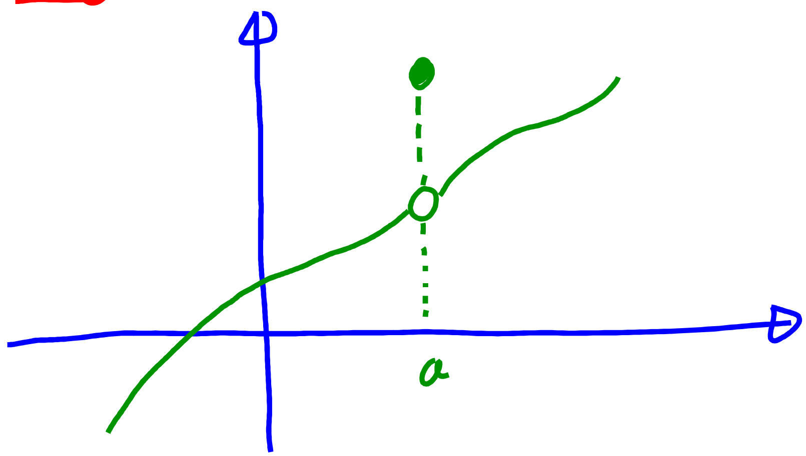
f is cont. on $[0, 1]$

$f(0) = 1$

$f(1) = -2$

By I.V.T, f has a zero in $[0, 1]$.

Types of Discontinuity



Removable discontinuity

f has a removable discontinuity at $x = a$ if

$\lim_{x \to a} f(x)$ exists but

$$\lim_{x \to a} f(x) \neq f(a)$$

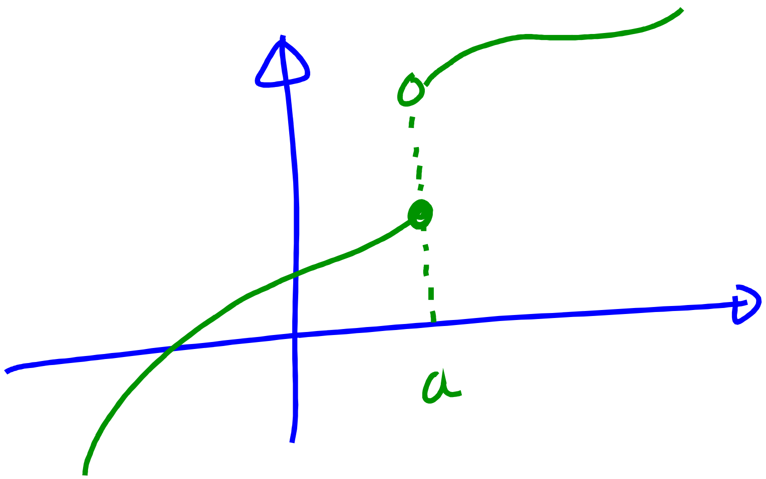
E.g.

$$f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2} & \text{if } x \neq -2 \\ k & \text{if } x = -2 \end{cases}$$

Find k such that f is continuous at $x = -2$.

$$\boxed{k = -1}$$

② Jump discontinuity



We say that f has a jump discontinuity at $x=a$ if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

E.g.

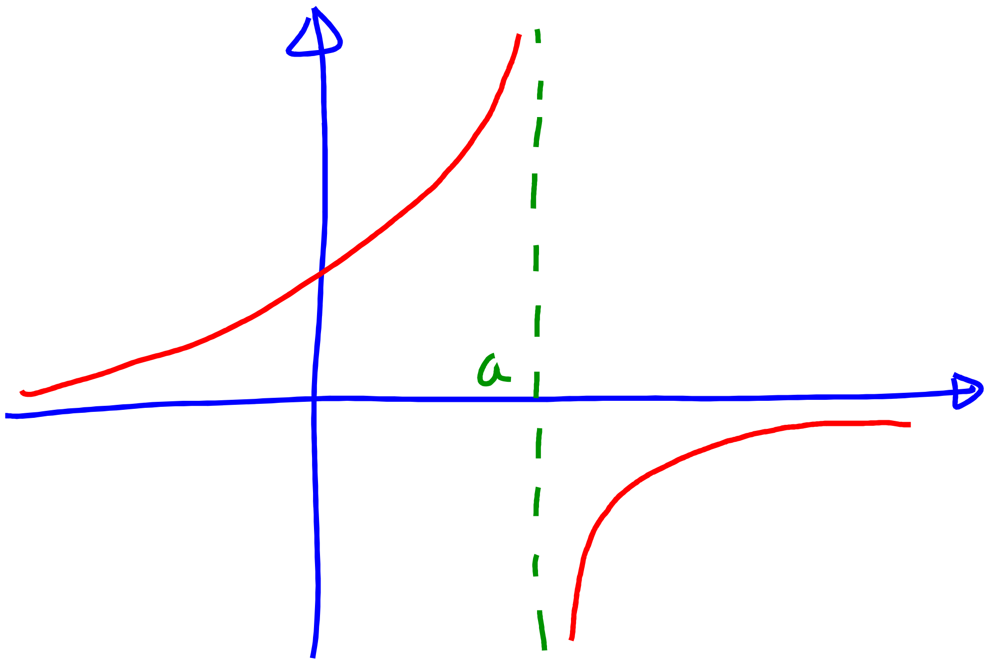
$$f(x) = \begin{cases} x\sin(x) & \text{if } x < \pi \\ x\cos(x) & \text{if } x \geq \pi. \end{cases}$$

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi} x\sin(x) = \pi\cdot\sin(\pi) = 0$$

$$\neq$$

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi} x\cos(x) = \pi\cos(\pi) = -\pi$$

③ Infinite Discontinuity



If $\lim_{x \to a^-} f(x) = -\infty$ or ∞

or if $\lim_{x \to a^+} f(x) = -\infty$ or ∞

then f has an infinite discontinuity at $x = a$.

<u>E.g.</u> $f(x) = \frac{1}{x}$

Inf. Discont. at $x = 0$