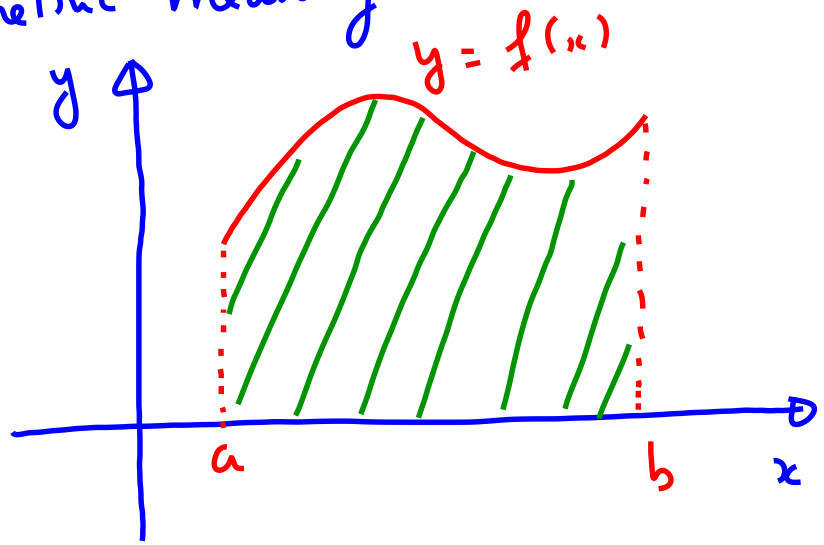
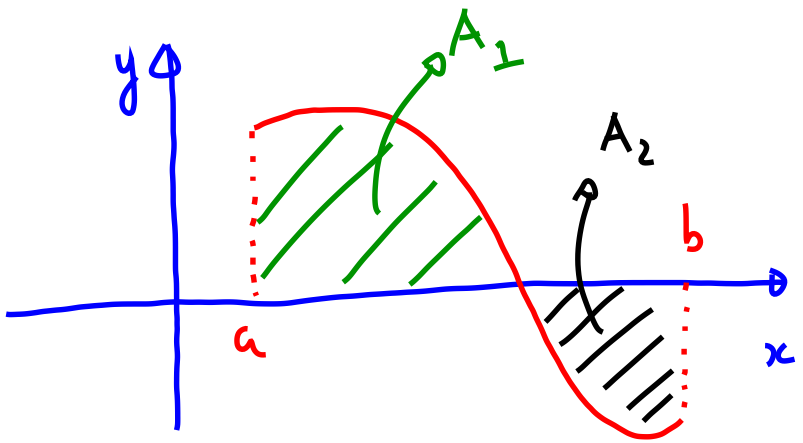
Definite Integrals and the Fundamental Theorem of Calculus.

Geometric meaning

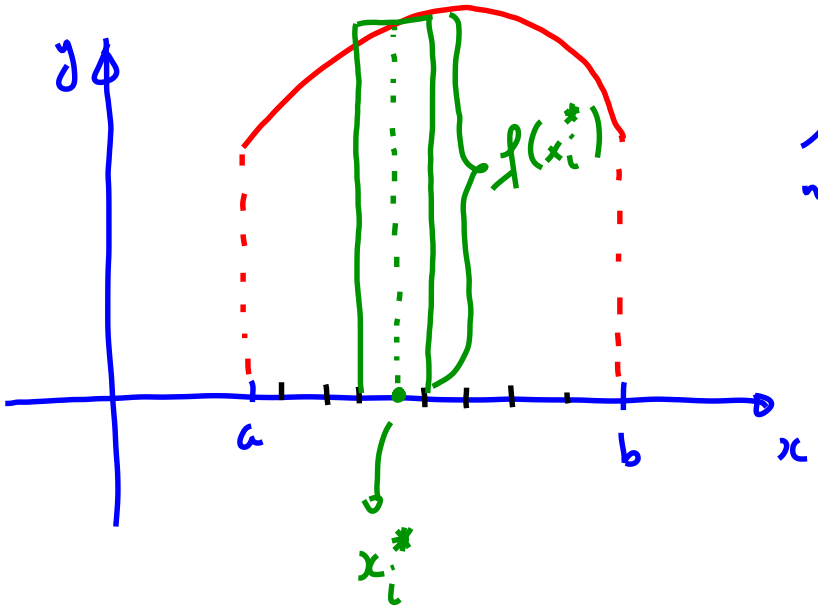


$$\int_a^b f(x)\,dx = \text{area under } y = f(x), \quad a \le x \le b.$$



$$\int_a^b f(x)\,dx = \text{signed area} = A_1 - A_2$$

Definition of definite integral



$$\Delta x = \frac{b-a}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus

Part II: If f is continuous on $[a,b]$ and $F(x)$ is an antiderivative of f on $[a,b]$ (i.e., $F'(x) = f(x)$)

then $\int_a^b f(x)\,dx = F(b) - F(a)$

Useful notation: $F(x)\Big|_a^b = F(b) - F(a)$.

$\int_a^b f(x)\,dx = F(x)\Big|_a^b$: this is read as $F(x)$ evaluated from a to b.

E.g. (1) $\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{(4)^3}{3} - \frac{(0)^3}{3} = \frac{64}{3}$.

② $\int_1^4 \frac{4}{x^2}\,dx = 4\cdot\int_1^4 x^{-2}\,dx = 4\cdot\frac{x^{-1}}{-1}\Big|_1^4 = \frac{-4}{x}\Big|_1^4$

$= \frac{-4}{4} - \frac{-4}{1} = -1 + 4 = 3$

Ex.

③ $\int_4^8 \left(4t^{\frac{5}{2}} - 3t^{\frac{3}{2}}\right) dt$

④ $\int_1^4 \frac{2-\sqrt{3}}{3^2}\,dz$

⑤ $\int_0^{\pi/2} \sin(\theta)\,d\theta$

⑥ $\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$

⑦ $\int_0^5 \sqrt{25-x^2}\,dx$

FTC, part I

Suppose $f(t)$ is a continuous function on $[a, b]$.

Define a function $F(x)$ by the following integral

$$F(x) = \int_a^x f(t)\,dt$$

Then FTC part I says that

$$F'(x) = f(x) \text{ for every } x \text{ in } [a, b].$$

In short,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

E.g.

$$\frac{d}{dx}\left(\int_2^x e^{-t^2}\,dt\right) = e^{-x^2}$$

$$\frac{d}{dr}\left(\int_4^r \frac{ds}{\sqrt{16-s^2}}\right) = \frac{1}{\sqrt{16-r^2}}$$

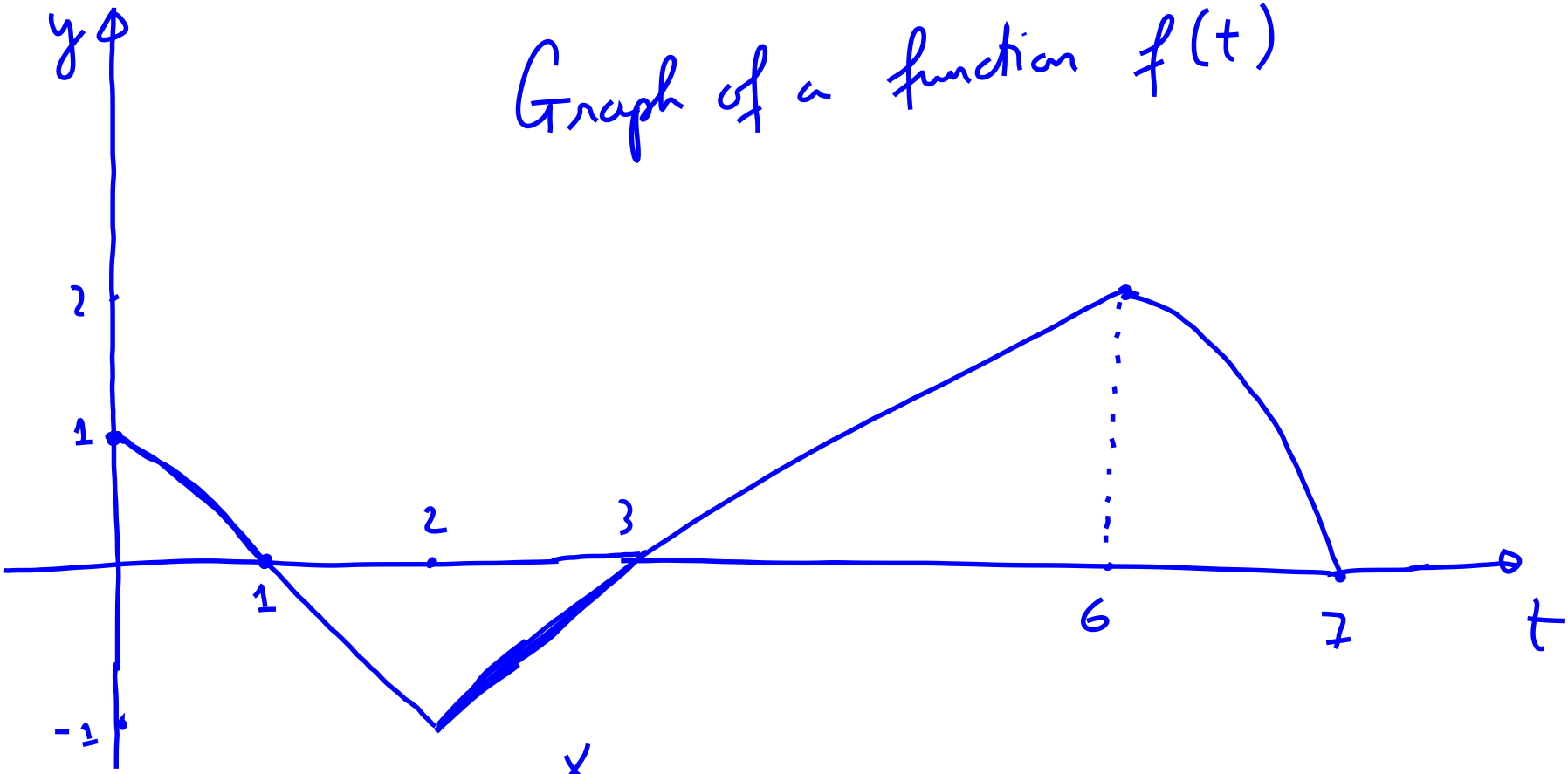
E.g. $\frac{d}{dx}\left(\int_1^{x^4} \sec(t)\,dt\right) = \sec(x^4)\cdot 4x^3$

(By the Chain Rule)

In general,

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = f(g(x))\cdot g'(x)$$

Graph of a function $f(t)$



Let $F(x) = \int_0^x f(t)\,dt$; $0 \le x \le 7$

① Find $F(0)$, $F(1)$, $F(2)$, $F(3)$, $F(6)$

② On which intervals is F increasing / decreasing

③ Find local min / max of F (if any) ?