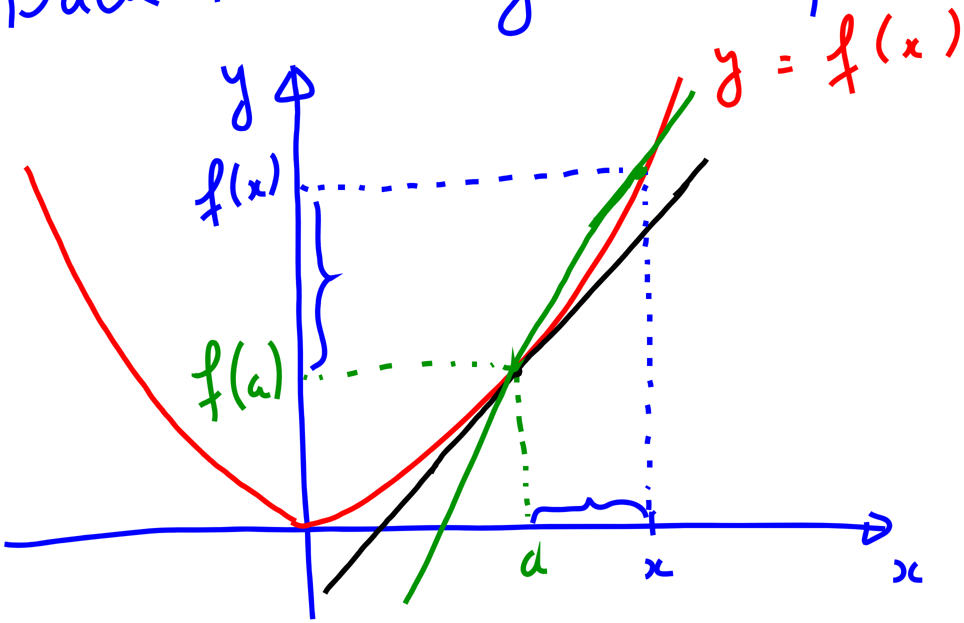
The Definition of the derivative

Back to the tangent line problem



Slope of the secant line through $(a, f(a))$ and $(x, f(x))$

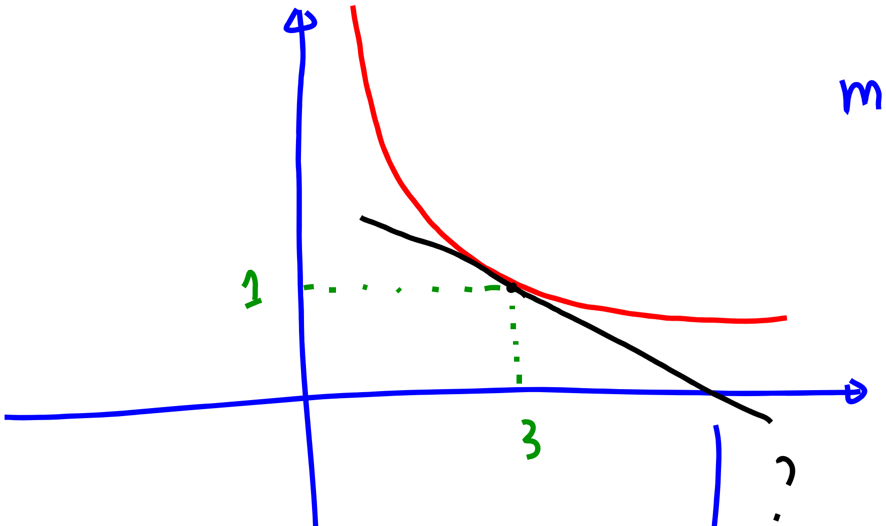
$$\frac{f(x) - f(a)}{x - a}$$

Slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$\boxed{m_{tangent} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided that limit exists.

E.g. $f(x) = \frac{3}{x}$. Find the slope and the equation of the tangent line to the graph of f at $(3,1)$



$$m_{tangent} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3} \frac{\frac{3}{x} - 1}{x - 3}$$

$$= \lim_{x \to 3} \frac{\boxed{\frac{3 - x}{x}}}{x - 3} =$$

$$= \lim_{x \to 3} \frac{-1\cancel{(3 - x)}}{x} \cdot \frac{1}{\cancel{x - 3}} = \lim_{x \to 3} \frac{-1}{x}$$

$$= \boxed{-\frac{1}{3}}$$ slope of tangent line.

Pt: $(3,1)$

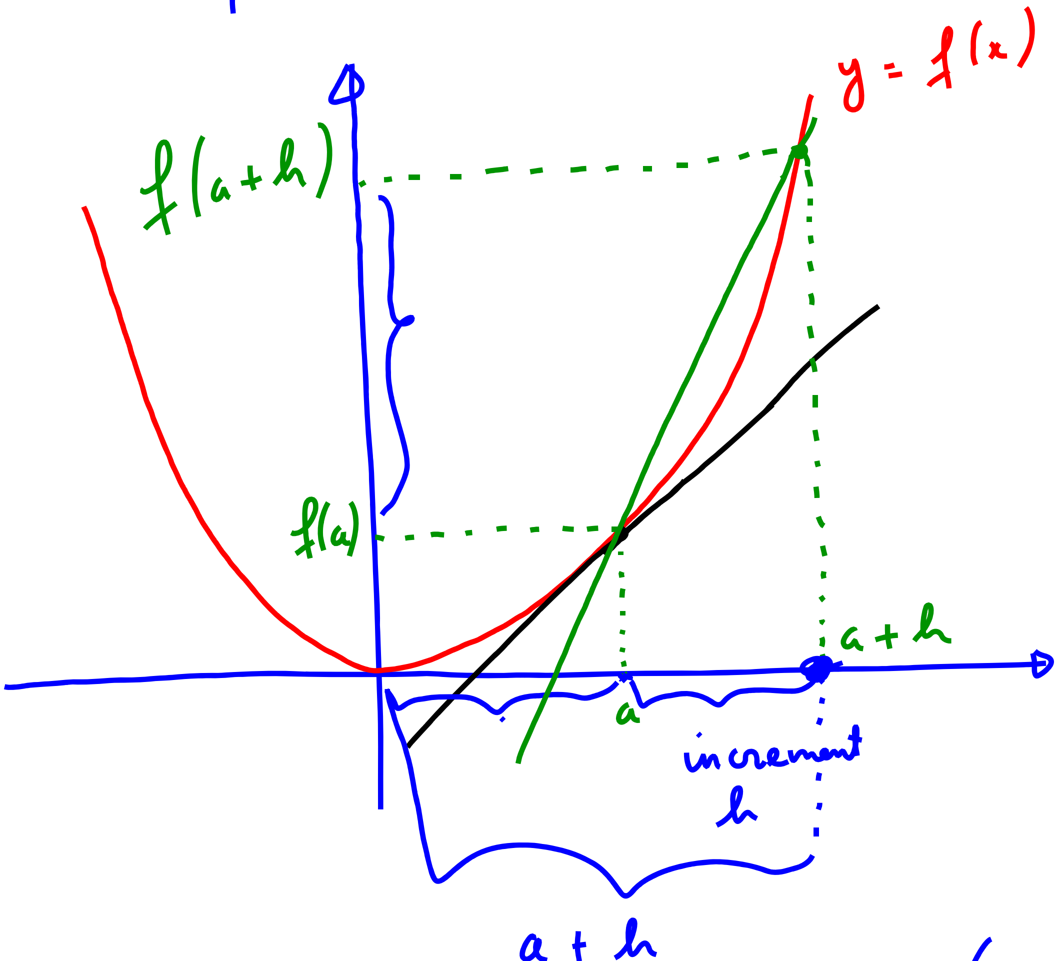
Point-Slope equation:

$$y - 1 = -\frac{1}{3}(x - 3)$$

Slope-intercept:

$$\boxed{y = -\frac{1}{3}x + 2}$$

An important variation of the formula for m_{tangent}.



Slope of secant line through $(a, f(a))$; $(a+h, f(a+h))$

$$\frac{f(a+h) - f(a)}{\cancel{a} + h - \cancel{a}}$$

$$\frac{f(a+h) - f(a)}{h}$$

The slope of tangent line at $(a, f(a))$ is given by

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

The slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ can calculated by:

$$\boxed{m_{tangent} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

E.g. Find the slope of the tangent line to the graph of $y = \sqrt{x}$ at $(1, 1)$ using the second formula.

$$m_{tangent} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\boxed{\sqrt{1+h} - 1}}{h} \cdot \frac{\boxed{\sqrt{1+h} + 1}}{\sqrt{1+h} + 1} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h} \cdot \left(\sqrt{1+h} + 1\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \boxed{\frac{1}{2}}.$$

* If $s = f(t)$: the position of an object at time t.

then $\lim_{t \to a} \boxed{\frac{f(t) - f(a)}{t - a}} = \lim_{h \to 0} \boxed{\frac{f(a+h) - f(a)}{h}}$

= instantaneous velocity at time $t = a$

average velocity over $[a, t]$

average velocity over $[\boxed{a}, a + h]$

* If $C = f(x)$ is the cost function, the cost of producing x units

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

average cost $[\boxed{a}, x]$

average cost over $[a, a+h]$

So, $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ gives us the marginal cost at the production level a; i.e., the rate at which the cost increases per unit when we are at the production level a.

In general, $y = f(x)$ describes a quantity

$$\underbrace{\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}}_{\text{Average Rate of change of that quantity over } [a, x]} \longrightarrow \text{instantaneous R.O.C of that quantity at a.}$$

Def: The derivative of $y = f(x)$ at the point $x = a$; denoted by $f'(a)$ (read as f prime of a) or $\left.\frac{dy}{dx}\right|_{x=a}$ (leibnitz notation), is defined to be

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

provided that the limit exists

Note: $f'(a)$ = slope of tangent line of $y=f(x)$ at $x=a$

$f'(a)$ = inst. velocity at $t=a$ if $y=f(t)$ is a position function.

$f'(a)$ = inst. r.o.c. at $x=a$ of $y=f(x)$

E.g. (1) $f(x)=\dfrac{1}{x^2}$. Find $f'(2)=\,?$

(2) If a rock is thrown upward on Mars with a velocity of 10 m/s; then its height after t seconds is given by

$$H(t)=10t-1.86t^2$$

(a) When will the rock hit the surface?

(b) What is the velocity of the rock when it hits the surface?

1) $\boxed{f(x) = \frac{1}{x^2}}$. $f'(2) = ?$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4-[4+4h+h^2]}{4(2+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{\cancel{4}-\cancel{4}-4h-h^2}{4(2+h)^2}}{h}$$

$$\lim_{h\to 0} \frac{\boxed{-4h-h^2}}{4(2+h)^2 \cdot h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(-4-h)}{4(2+h)^2 \cdot \cancel{h}}$$

$$\lim_{h\to 0} \frac{-4-h}{4(2+h)^2}$$

$$= \frac{-4}{16} = \boxed{-\frac{1}{4}}.$$

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$$H(t) = 10t - 1.86t^2 = 0$$

$$t \cdot (10 - 1.86t) = 0$$

$t = 0$ or $10 - 1.86t = 0$

$$\boxed{t = 5.38}$$

$$H'(a) = \lim_{t \to a} \frac{H(t) - H(a)}{t - a}$$

$$= \lim_{t \to a} \frac{10t - 1.86t^2 - (10a - 1.86a^2)}{t - a}$$

$$= \lim_{t \to a} \frac{10t - 1.86t^2 - 10a + 1.86a^2}{t - a}$$

$$= \lim_{t \to a} \frac{10(t-a) + 1.86(a^2 - t^2)}{t - a}$$

$$= \lim_{t \to a} \frac{10(t-a) + 1.86(a-t)(a+t)}{t - a}$$

$$= \lim_{t \to a} \frac{10\boxed{(t-a)} - 1.86\boxed{(t-a)}(a+t)}{t - a}$$

$$\lim_{t \to a} \frac{\cancel{(t-a)}\left[10 - 1.86(a+t)\right]}{\cancel{t-a}}$$

$$\lim_{t \to a} \left[10 - 1.86(a+t)\right] = 10 - 1.86(2a)$$

$$\boxed{H'(a) = 10 - 3.72a.}$$

$$H'(5.38) = 10 - 3.72 \cdot (5.38) = \boxed{-10.0136}$$