Derivatives of Inverse Functions

We know how to find the derivative of

* Polynomial Functions
* Rational functions
* Trig Functions
* Composition of these functions

What about the derivative of arcsin, arccos, arctan?

<u>Recall</u>: $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$; $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

$\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

$\arctan(1) = \frac{\pi}{4}$.

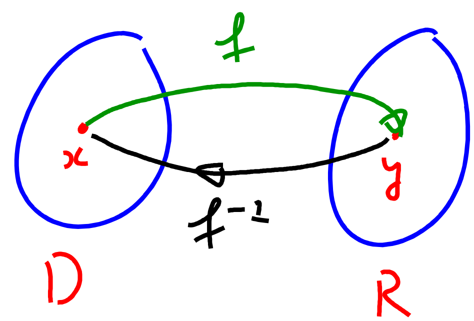
$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

arcsin(x) gives the angle y ; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ such that $\sin y = x$

arccos(x) ———— ; $0 \le y \le \pi$ such that $\cos y = x$

arctan(x) ———— ; $-\frac{\pi}{2} < y < \frac{\pi}{2}$ ———— $\tan y = x$.

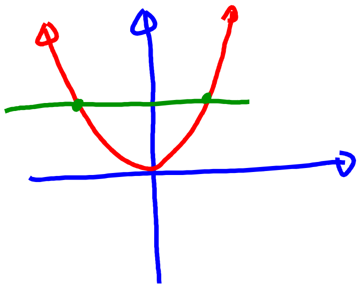
In general, if $f(x) = y$ then the inverse function of f, denoted by f^{-1} $\left(\text{not equal to } \frac{1}{f}\right)$ is the function that undoes what f does. $f^{-1}(y) = x$

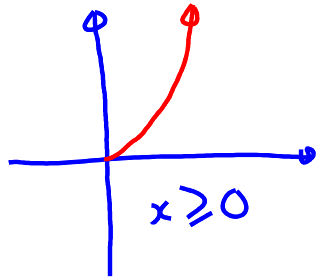


$$\boxed{\begin{aligned} f^{-1}(f(x)) &= x \\ f(f^{-1}(x)) &= x \end{aligned}}$$

<u>E.g.</u> $f(x) = x+2$; $f^{-1}(x) \overset{?}{=} x - 2$.

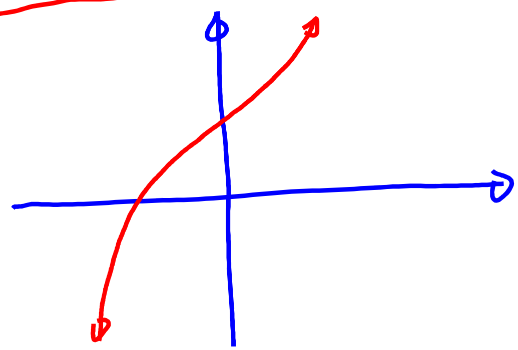
$f(x)=x^2$; $f^{-1}(x)=\sqrt{x}$





$x^3+2x+3=y$

$\boxed{f(x)=x^3+2x+3}$; $f^{-1}(x) \cancel{\stackrel{?}{=}} \dfrac{\sqrt[3]{x}}{2}-3$



$f^{-1}(f(x))=x$

$f^{-1}(x^3+2x+3)$

$$=\frac{\sqrt[3]{x^3+2x+3}}{2}-3 \stackrel{?}{=} x$$

Inverse Function Theorem:

Let f be a function that is invertible and differentiable on an interval containing x.

Then:

$$\boxed{\underbrace{\frac{d}{dx}\left(f^{-1}(x)\right)}_{(f^{-1})'(x)} = \frac{1}{f'\left(f^{-1}(x)\right)}; \quad \text{provided that } f'\left(f^{-1}(x)\right) \neq 0}$$

If we call f^{-1} as g, then $g'(x) = \dfrac{1}{f'(g(x))}$

E.g. let $f(x) = x^5 + 3x^3 - 4x - 8$

$f(1) = -8$. So, $f^{-1}(-8) = 1$

So, $(-8, 1)$ belongs to the graph of f^{-1}.

Q: Find the equation of the tangent line to the graph of f^{-1} at the point $(-8, 1)$.

$\rightarrow$ Find $(f^{-1})'(-8) = \dfrac{1}{f'(f^{-1}(-8))} = \dfrac{1}{f'(1)}$

$f'(x) = 5x^4 + 9x^2 - 4$. $f'(1) = 10$

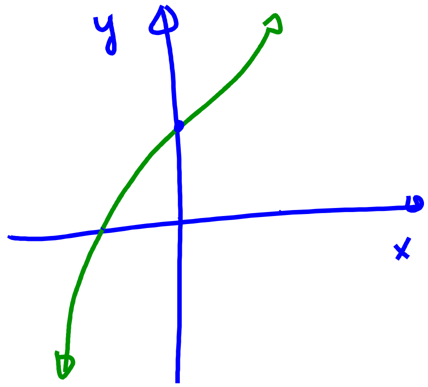
$(f^{-1})'(-8) = \dfrac{1}{10} \leftarrow$ slope.

Point-Slope equation: $y - 1 = \dfrac{1}{10}(x - (-8))$

$$\boxed{y = \frac{x}{10} + \frac{9}{5}}$$

Ex. let $f(x) = x^3 + 2x + 3$

Find $(f^{-1})'(0)$.



$$(f^{-1})'(0) = \frac{1}{f'(\boxed{f^{-1}(0)})}$$

Find $f^{-1}(0)$.

$f^{-1}(0) = ?$

$$\boxed{f(0) = 3 \to f^{-1}(3) = 0}$$

$f(?) = 0$

$f^{-1}(0)$

$(?)^3 + 2(?) + 3 = 0$

$x^3 + 2x + 3 = 0$

$x = -1$ satisfies this equation.

$f(-1) = 0$; $f^{-1}(0) = -1$

$f'(x) = 3x^2 + 2$. $f'(-1) = 5$.

$$(f^{-1})'(0) = \frac{1}{f'(-1)} = \boxed{\frac{1}{5}}$$

Rational Zero Theorem:

The rational zero of $x^3 + 2x + 3$ has the form $\frac{p}{q}$ where p is a factor of 3; q is a factor of 1

$$\frac{p}{q} = \pm 3;\ \pm 1.$$

Proof of I.F.T.

Key: $f(f^{-1}(x)) = x$.

Take the derivative w.r.t. x of both sides:

$$\frac{d}{dx}\left(f(f^{-1}(x))\right) = \frac{d}{dx}(x) = 1$$

By Chain Rule:

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Derivatives of Inverse Trig Functions

① $$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\underbrace{\hphantom{(\arcsin x)}}$$

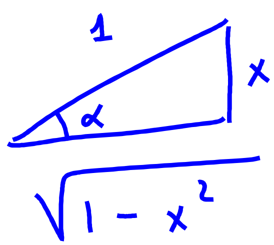
$$\frac{d}{dx}\left(\sin^{-1} x\right)$$

$f(x) = \sin^{-1} x$

$g(x) = \sin x$

By IFT. $$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

$\sin^{-1} x = \alpha \Rightarrow \sin\alpha = x$



$\Rightarrow \cos\alpha = \sqrt{1-x^2}$

$$= \frac{1}{\sqrt{1-x^2}}$$

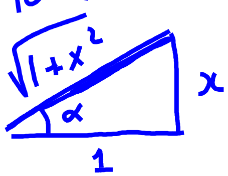
(2) $$\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}.$$

(3) $$\underbrace{\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}.}_{\text{Prove this!}}$$

$$f(x) = \tan^{-1}x \;;\quad g(x) = \tan x \;;\quad g'(x) = \sec^2 x$$

Want: $$f'(x) \overset{?}{=} \frac{1}{g'(f(x))} = \frac{1}{\sec^2\left(\tan^{-1}x\right)}.$$

$$\tan^{-1}x = \alpha \;;\quad \tan\alpha = x$$



$$;\quad \cos\alpha = \frac{1}{\sqrt{1+x^2}} \;;\quad \sec\alpha = \sqrt{1+x^2} \;;\quad \sec^2\alpha = 1+x^2$$

(For rest of the formula: look up in book).

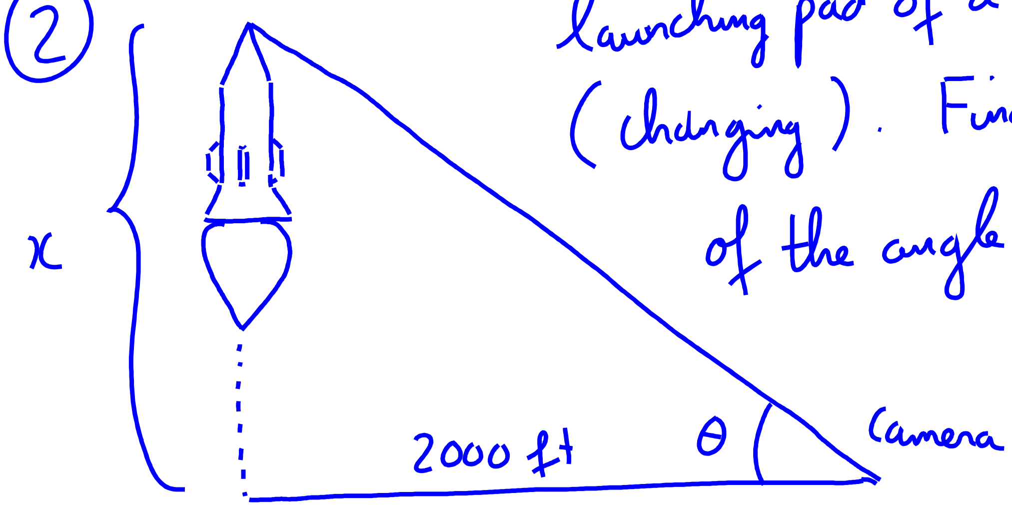
Ex. (1) Find $\frac{dy}{dx}$.

(a) $y = \tan^{-1}(x^2)$

(b) $y = \cos^{-1}(3x-1)$

(c) $y = x^2 \sin^{-1} x$.

(2)



Television camera is 2000 ft away from the launching pad of a rocket. x: height of rocket (changing). Find the rate of change $\frac{d\theta}{dx}$ of the angle of elevation θ of the camera when the rocket is 5000 ft away from camera.