Derivatives of Exponential and Logarithmic Functions

u is a function of x

$$\frac{d}{dx}\left(x^n\right) = n x^{n-1}$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(u^n\right) = n u^{n-1} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\sin u\right) = \cos u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos u\right) = -\sin u \, \frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan u\right) = \sec^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

1) Derivatives of Exponential functions

If $f(x) = e^x$, then $f'(x) = e^x$

In leibnitz notation: $\boxed{\frac{d}{dx}(e^x) = e^x}$

If u is a function of x, $\boxed{\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}}$

In Newton's notation: $\boxed{\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)}$

Ex. Find the derivative:

a) $\underbrace{\frac{d}{dx}\left(e^{\pi}\right)}$

0

b) $\underbrace{\frac{d}{dx}\left(e^{-x}\right)}$

$-e^{-x}$

c) $\underbrace{\frac{d}{dx}\left(e^{\sec x}\right)}$

$e^{\sec x} \cdot \sec x \tan x$

d) $\underbrace{\frac{d}{dx}\left(x e^{2x}\right)}$

$e^{2x} + 2xe^{2x}$

$e^{2x}(1+2x)$

Bases other than e.

$a > 0$, $a \neq 1$. $a \neq e$

Then $\boxed{\frac{d}{dx}\left(a^x\right) = a^x \cdot \ln a}$ $\boxed{\frac{d}{dx}(a^u) = a^u \cdot \ln a \cdot \frac{du}{dx}}$

(Prf: $a^x = e^{x \ln a}$. $\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x \ln a}\right)$

$= e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$.)

Ex: (a) $\frac{d}{dx}\left(5^{\cos(3x)}\right)$; (b) $\frac{d}{dx}\left(x^{\pi} \cdot \pi^{x}\right)$

$\underbrace{\frac{d}{dx}\left(5^{\cos(3x)}\right)}_{5^{\cos(3x)} \cdot \ln 5 \cdot (-\sin(3x)) \cdot 3}$

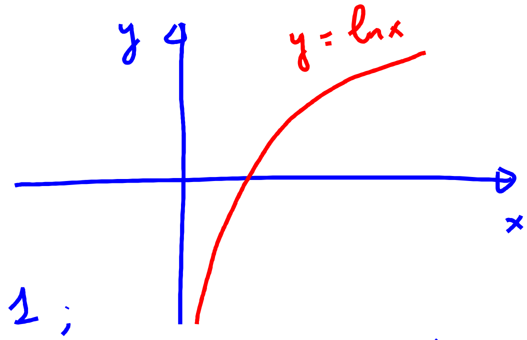
$\pi x^{\pi - 1} \cdot \pi^x +$

$x^{\pi} \cdot \pi^x \cdot \ln \pi$

$x^{\pi-1} \cdot \pi^x (\pi + x \ln \pi)$

Derivatives of Log functions:

Natural log function: $y = \ln x$



$$\left(\ln 1 = 0\,;\ \ln e = 1\,;\ \ln e^2 = 2\,;\ \ln e^{\alpha} = \alpha \right)$$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad ; \quad \frac{d}{dx}\left(\ln u\right) = \frac{1}{u}\cdot\frac{du}{dx}}$$

Prf: $y = \ln x \iff e^{y} = x$. Take derivative w.r.t. x of both sides:

$$e^{y}\cdot\frac{dy}{dx} = 1\ ; \quad \frac{dy}{dx} = \frac{1}{e^{y}} = \frac{1}{x}.$$

Useful properties of the natural log function

$$\boxed{\begin{aligned} &(1)\quad \ln(uv) = \ln u + \ln v \\ &(2)\quad \ln(u^p) = p \ln u \\ &(3)\quad \ln\left(\frac{u}{v}\right) = \ln u - \ln v \end{aligned}}$$

E.g. $\frac{d}{dx}\left(\ln\left((3x+2)^5\right)\right) = \frac{d}{dx}\left(5 \ln(3x+2)\right)$

$= 5 \frac{d}{dx}\left(\ln(3x+2)\right) = 5 \cdot \frac{1}{3x+2} \cdot 3 = \boxed{\frac{15}{3x+2}}$

Ex: (a) $\frac{d}{dx}\left(\ln(\sin x)\right)$ (b) $\frac{d}{dx}\left(\ln\sqrt{5x-7}\right)$

(c) $\frac{d}{dx}\left(\ln\left[\frac{(2x+1)^3}{\sqrt{x^2+1}}\right]\right)$

Base other than e.

$$y = \log_a x$$

$$\boxed{\begin{aligned} \frac{d}{dx}\left(\log_a x\right) &= \frac{1}{x\ln a}. \\ \frac{d}{dx}\left(\log_a u\right) &= \frac{1}{u\ln a}\cdot\frac{du}{dx} \end{aligned}}$$

* $y = x^x$; $\frac{dy}{dx} = ?$

$$y = x^x = e^{x \ln x}$$

$$\frac{dy}{dx} = e^{x \ln x} \cdot \frac{d}{dx}(x \ln x) = e^{x \ln x} \cdot \left(\ln x + 1\right)$$

$$= \boxed{x^x \cdot (\ln x + 1)}$$

We can now take the derivative of any function of the form $y = f(x)^{g(x)}$

$$y = f(x)^{g(x)} = e^{g(x)\cdot\ln(f(x))}$$

$$\frac{dy}{dx} = e^{g(x)\cdot\ln(f(x))} \cdot \frac{d}{dx}\left[g(x)\cdot\ln(f(x))\right]$$

$$= f(x)^{g(x)} \cdot \left[g'(x)\ln(f(x)) + g(x)\cdot\frac{1}{f(x)}\cdot f'(x)\right]$$

$$= f(x)^{g(x)} \cdot \left[g'(x)\ln(f(x)) + \frac{g(x)\cdot f'(x)}{f(x)}\right].$$