# 3.3. Differentiation Rules

$$\underbrace{f'(x)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx}$$

$f(x) = x^2$ ; $f'(x) = 2x$

$f(x) = x^3$ ; $f'(x) = 3x^2$

# Differentiation Rules

1. Power Rule
2. Sum and Difference Rule
3. Product and Quotient Rule.

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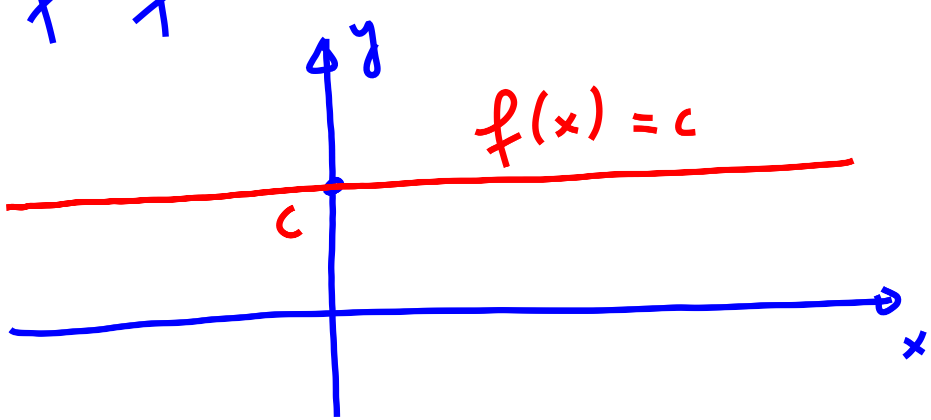
* Derivative of a constant function

If $f(x) = c$ ; $c$ : constant ; then $f'(x) = 0$.




$$\boxed{\frac{d}{dx}[c] = 0}$$

② Power Rule.

$$\frac{d}{dx}\left(x^{\boxed{2}}\right) = 2x \ , \quad \frac{d}{dx}\left(x^3\right) = 3x^2.$$

$$\frac{d}{dx}\left(x^4\right) = 4x^3 \ ; \quad \frac{d}{dx}\left(x^5\right) = 5x^4$$

In general, $\boxed{\frac{d}{dx}\left[x^n\right] = n x^{n-1}}$

$n$ : a positive integer

If $f(x) = x^n$, then $f'(x) = n x^{n-1}$.

If $r$ is any real #, then $\boxed{\frac{d}{dx}\left[x^r\right] = r x^{r-1}}$

$$\frac{d}{dx}\left[x^{2017}\right] = 2017x^{2016}$$

$$\frac{d}{dx}\left[\boxed{e^{2017}}\right] = 0$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

$$\boxed{\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -1 \cdot x^{-2} = \frac{-1}{x^2}$$

$$\boxed{\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}}$$

$$\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x^2}}\right] = \frac{d}{dx}\left[\frac{1}{x^{2/3}}\right] = \frac{d}{dx}\left[x^{-2/3}\right]$$

$$= -\frac{2}{3} \cdot x^{-\frac{2}{3}-1} = -\frac{2}{3} x^{-\frac{5}{3}}$$

③ Sum, difference and the constant multiple rule.

Sum: $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

In prime notation: $\left[f(x) + g(x)\right]' = f'(x) + g'(x)$

E.g. $\frac{d}{dx}\left[x^3 + \frac{1}{x}\right] = 3x^2 - \frac{1}{x^2}$.

Difference rule: $\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$

$\left[f(x) - g(x)\right]' = f'(x) - g'(x)$

Constant multiple rule: $\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$

$(c\, f(x))' = c\, f'(x)$

E.g. $\frac{d}{dx}\left[\pi \cdot \sqrt{x}\right] = \pi \cdot \frac{d}{dx}\left[\sqrt{x}\right] = \pi \cdot \frac{1}{2\sqrt{x}} = \frac{\pi}{2\sqrt{x}}$.

E.g. $f(x) = x^3 - 4x^2 + 3x + 6$

$f'(x) = 3x^2 - 8x + 3$.

Find the equation of the tangent line to the graph of $y = f(x)$ at $(0,6)$.

Slope $= f'(0) = 3$

Point-Slope equation: $y - 6 = 3x$

$y = 3x + 6$.

Find all points on the graph where the tangent line is horizontal.

$$f'(x) = 3x^2 - 8x + 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 3}}{6} = \frac{8 \pm \sqrt{28}}{6}$$

# (4) Product and Quotient Rule

$$\cancel{\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right]\cdot\frac{d}{dx}\left[g(x)\right]}$$

this is wrong.

Eg.

$$\underbrace{\frac{d}{dx}\left[x^2\cdot x^3\right]}_{\displaystyle \frac{d}{dx}\left[x^5\right] = 5x^4} = \underbrace{\underbrace{\frac{d}{dx}\left[x^2\right]}_{2x}\cdot\underbrace{\frac{d}{dx}\left[x^3\right]}_{3x^2}}_{6x^3}$$

$$5x^4 \neq 6x^3$$

Correct formula:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right]\cdot g(x) + f(x)\cdot\frac{d}{dx}\left[g(x)\right]$$

Prime notation.

$$\boxed{\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

E.g. $j(x) = (2x^5 - 1)\cdot(x^2 + x)$

$j'(x) = ?$

$j(x) = 2x^7 + 2x^6 - x^2 - x.$ $\boxed{j'(x) = 14x^6 + 12x^5 - 2x - 1}$

$$j(x) = \underbrace{\boxed{(2x^5 - 1)}}_{f(x)}\underbrace{\boxed{(x^2 + x)}}_{g(x)}$$

$$= \underbrace{10x^4}_{f'(x)} \cdot \underset{g(x)}{(x^2 + x)} + \underbrace{(2x+1)}_{g'(x)} \cdot \underbrace{(2x^5 - 1)}_{f(x)}$$

Check: $10x^6 + 10x^5 + 4x^6 + 2x^5 - 2x - 1$

$$14x^6 + 12x^5 - 2x - 1$$

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Quotient Rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2}.$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

high

low

$$\frac{\text{low } d(\text{high}) - \text{high } d(\text{low})}{(\text{low})^2}$$

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Eg. Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}$.

$$h'(x) = \frac{3(4x-3) - 4(3x+1)}{(4x-3)^2} = \frac{12x-9-12x-4}{(4x-3)^2}$$

$$= \frac{-13}{(4x-3)^2}$$

Ex. Find the derivative of the given function.

ⓐ $f(x) = x^2 \cdot \left(\frac{2}{x^2} + \frac{5}{x^3}\right)$

$f'(x) = -\frac{5}{x^2}$

$f(x) = 2 + \frac{5}{x}$

ⓑ $g(x) = \frac{x^2+4}{x^2-4}$

$g'(x) = \frac{-16x}{(x^2-4)^2}$.

ⓒ $h(x) = \sqrt{x} \cdot \left(\frac{1}{x} + x^3\right)$

$f(x) = 2 + \frac{5}{x}$; $f'(x) = \frac{-5}{x^2}$.

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(c) $h(x) = \sqrt{x} \cdot \left( \frac{1}{x} + x^3 \right)$

$= x^{\frac{1}{2}} \cdot \left( x^{-1} + x^3 \right)$

$= x^{-1/2} + x^{7/2}$

$h'(x) = \boxed{-\frac{1}{2} x^{-3/2} + \frac{7}{2} x^{5/2}}$

Higher order derivative.

read as f double prime

$f(x) = x^3$; $f'(x) = \boxed{3x^2}$; $\boxed{f''(x)} = 6x$

Second derivative of f = derivative of f'

Third derivative: $f'''(x) = 6$

Fourth derivative: $f^{(4)}(x) = 0$

Fifth derivative: $f^{(5)}(x) = 0$

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In Leibniz notation: $\dfrac{dy}{dx}$

Second derivative: $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\boxed{d^2}y}{dx^2}$

differentiate twice

take derivative w.r.t.

Third derivative: $\dfrac{d^3y}{dx^3}$