How derivatives affect the shape of a graph

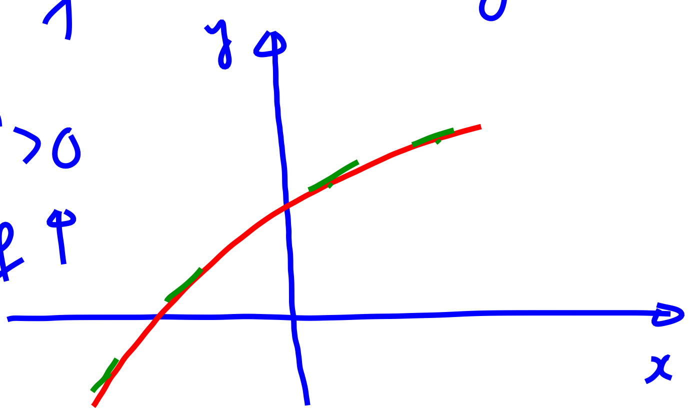
* **What does f' say about f?**

Increasing / Decreasing Test

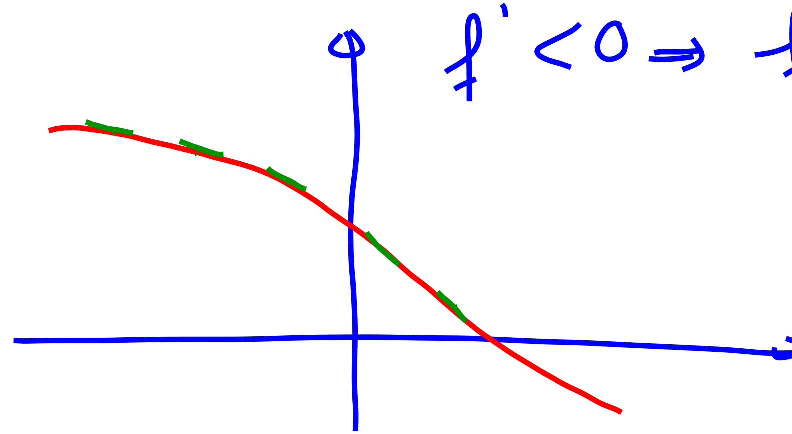
① If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.

② If $f'(x) < 0$ for all x in an interval I, then f is decreasing on I.

$f' > 0 \Rightarrow f \uparrow$

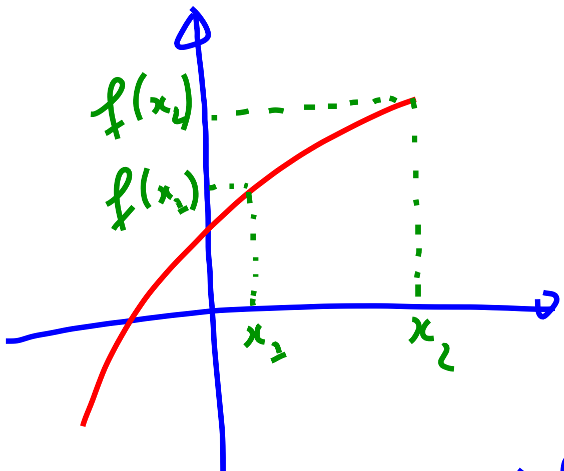


$f' < 0 \Rightarrow f \downarrow$



Proof:

① $\boxed{f' > 0 \text{ on } I} \implies f\uparrow$ on I.



Increasing means:

$$\boxed{x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)}$$

Given: $f' > 0$ on I.

Choose $x_1 < x_2$ and x_1 ; x_2 are in I.

Then we want to show that $f(x_1) < f(x_2)$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 \text{ for some } c \text{ in } (x_1, x_2)$$

by MVT.

So, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$

$x_2 - x_1 > 0$

$\Rightarrow f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_2) > f(x_1)$.

$\blacksquare$

<u>Eg.</u> $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

<u>Q</u>: Determine the interval on which f is increasing/decreasing.

$f'(x) = \underbrace{12x^3 - 12x^2 - 24x}$.

$f'(x) = 0$.

$12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$\boxed{12x(x-2)(x+1)} = 0$

$x = 0$; $x = 2$; $x = -1$ ⟵ critical points

x	$-\infty$		$\boxed{-1}$		$\boxed{0}$		$\boxed{2}$		∞
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\searrow$	local min @ $x=-1$	$\nearrow$	local max @ $x=0$	$\searrow$	local min @ $x=2$	$\nearrow$	

Decreasing on $(-\infty,-1)\cup(0,2)$

Increasing on $(-1,0)\cup(2,\infty)$

$\longrightarrow$ The first derivative test.

The first derivative test

If c is a critical point of a continuous function, then

① If f' changes from positive to negative at c, then f has a local max at c.

$f'>0 \quad | \quad f'<0$

$f \nearrow \; c \; \searrow$

local max

② If f' changes from negative to positive at c, then f has a local min at c.

$f'<0 \quad | \quad f'>0$

$\searrow \; c \; \nearrow$

local min

③ If f' does not change sign at c, then f has no local max nor min at c.

Ex. $g(x) = x + 2\sin x$; on $[0, 2\pi]$

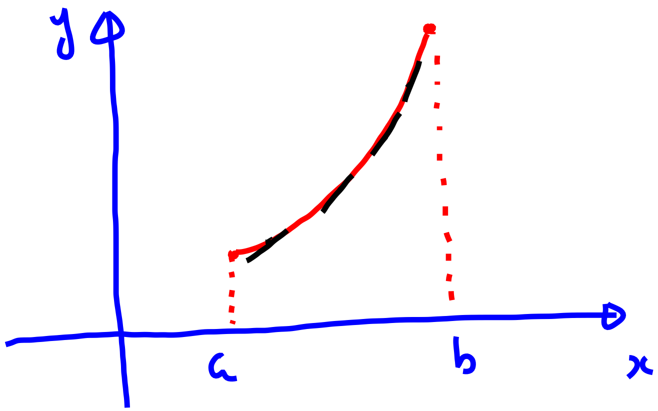
Q: Find the $\underbrace{\text{local max}}_{(x\text{-coord},\ y\text{-coord})}$ and local min of g on $[0, 2\pi]$

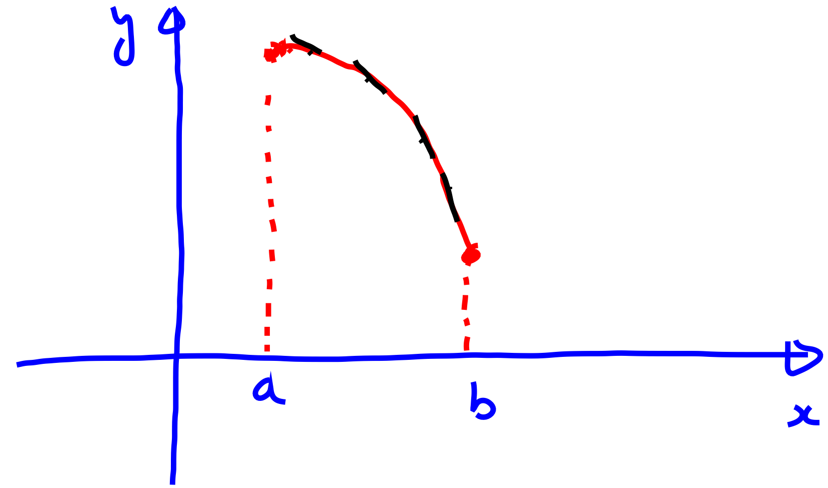
A: local max $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$

local min $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$

Ex. 1. Webassign.

What does f'' say about f ? $\longrightarrow$ concavity





Concave up; i.e., graph lies above tangent lines

Slopes are increasing, i.e., $f' \uparrow$; i.e., $f'' > 0$

Concave down; i.e., graph lies below tangent lines.

Slopes are decreasing. i.e., $f' \downarrow$; i.e., $f'' < 0$

Concavity Test

(a) If $f''(x) > 0$ on an interval I, then f is concave up on I.

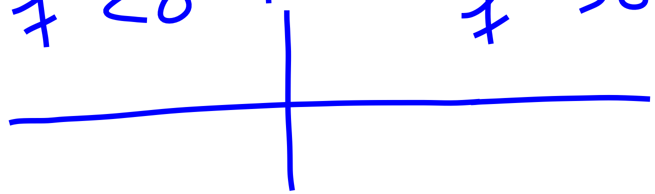
(b) If $f''(x) < 0$ on an interval I, then f is concave down on I.

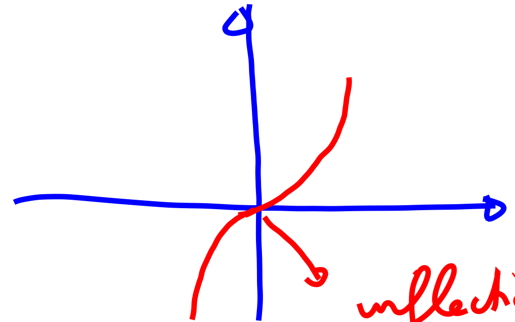
<u>Definition</u>: A point P is called an <u>inflection point</u> of a function $y = f(x)$ if the graph changes from being concave up to concave down at P or from concave down to concave up.

$f''>0 \quad f''(x)=0 \quad f''<0$ or $f''<0 \quad f''(c)=0 \quad f''>0$

$f(x) = x^3$; $f''(x) = 6x = 0$ when $x = 0$

$f'' < 0$ $\quad f''(0) = 0 \quad$ $f'' > 0$

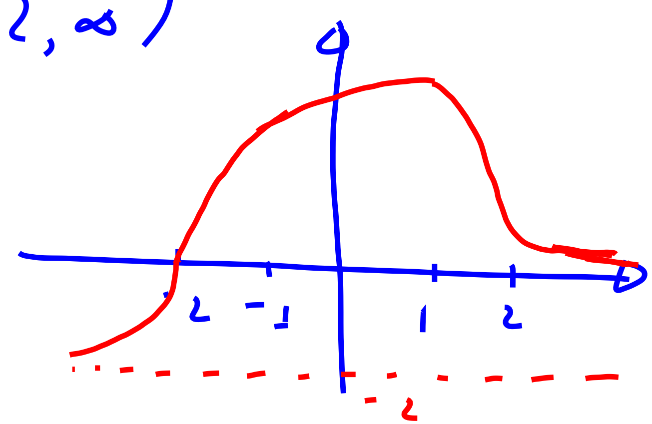




Ex. Sketch a possible graph of a function f that satisfies the following conditions

1. $f'(x) > 0$ on $(-\infty, 1)$; $f'(x) < 0$ on $(1, \infty)$
2. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$

   $f''(x) < 0$ on $(-2, 2)$
3. $\lim_{x \to -\infty} f(x) = -2$; $\lim_{x \to \infty} f(x) = 0$



Second Derivative test (E.g. Webassign HW #14)

If $f'(c) = 0$ and $f''(c) > 0$,
then f has a local min at c.

If $f'(c) = 0$ and $f''(c) < 0$,
then f has a local max at c.

If $f'(c) = 0$ and $f''(c) = 0$, then we
can say nothing.