# 4.8. L'Hopital Rule

## L'Hopital Rule for limits of the form $\frac{0}{0}$

E.g. $\displaystyle\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9} = \frac{0}{0}$

L'Hopital: $\displaystyle\lim_{x\to 3} \frac{2x-2}{2x} = \frac{4}{6} = \boxed{\frac{2}{3}}$.

( Previously, $\displaystyle\lim_{x\to 3} \frac{(x+1)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \lim_{x\to 3} \frac{x+1}{x+3} = \frac{4}{6} = \frac{2}{3}$. )

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L'Hopital Rule for $\frac{0}{0}$ limits

Suppose that we want to find $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ where

$\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$; that is, we have a $\frac{0}{0}$ limit.

+ Then L'Hopital rule says that

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

E.g.

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{-1}{2\sqrt{1-x}}}{1}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}}{1} = \frac{\frac{1}{2} + \frac{1}{2}}{1} = \boxed{1}.$$

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Ex.:

① $\displaystyle \lim_{x \to 0} \frac{e^x - x - 1}{x^2}$

② $\displaystyle \lim_{x \to 0} \frac{\tan x - x}{x^3}$

Apply L'Hopital Rule to find these limits.

1. $$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$$

2. $$\lim_{x\to 0} \frac{\tan x - x}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{2\sec x \cdot \sec x \cdot \tan x}{6x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \cdot \tan x}{6x} = \boxed{\lim_{x\to 0} \frac{2\sec^2 x}{6}} \cdot \boxed{\lim_{x\to 0} \frac{\tan x}{x}}$$

$$= \frac{1}{3} \cdot 1 = \boxed{\frac{1}{3}}$$

$$\frac{\tan x}{x} = \frac{\frac{\sin x}{\cos x}}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$\frac{0}{0}$ form is called an indeterminate form

Other indeterminate forms are

$\frac{\infty}{\infty}$

$\infty^0$

$1^\infty$

$0^0$

$\infty - \infty$

$0 \cdot \infty$

$\lim_{x \to \infty} \frac{\ln x}{x}$ ; $\lim_{x \to \infty} \frac{e^x}{x}$

$\lim_{x \to \infty} x^{\frac{1}{x}}$

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$\lim_{x \to 0} x^{\sqrt{x}}$

$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

$\lim_{x \to \infty} x^3 \cdot e^{-x^2}$

Strategy to find limits in indeterminate form.

1. If the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can immediately apply L'Hopital Rule.

2. If the form is $\infty \cdot 0$, the problem is like $\lim_{x \to a} f(x) \cdot g(x)$, we can rewrite this as $\lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}$ and turn it into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

③ If the limit is of the form $0^0$; $\infty^0$, $1^\infty$ the problem will be like

$$L = \lim_{x\to a} f(x)^{g(x)}.$$

Take ln: $\ln L = \lim_{x\to a} g(x) \cdot \ln(f(x))$

then this has the form $\infty \cdot 0$. Rewrite as in ②

④ Limit of the form $\infty - \infty$. Find common denominator, combine