2.3. Limit laws

$$\lim_{x \to a} f(x)$$

Numerically

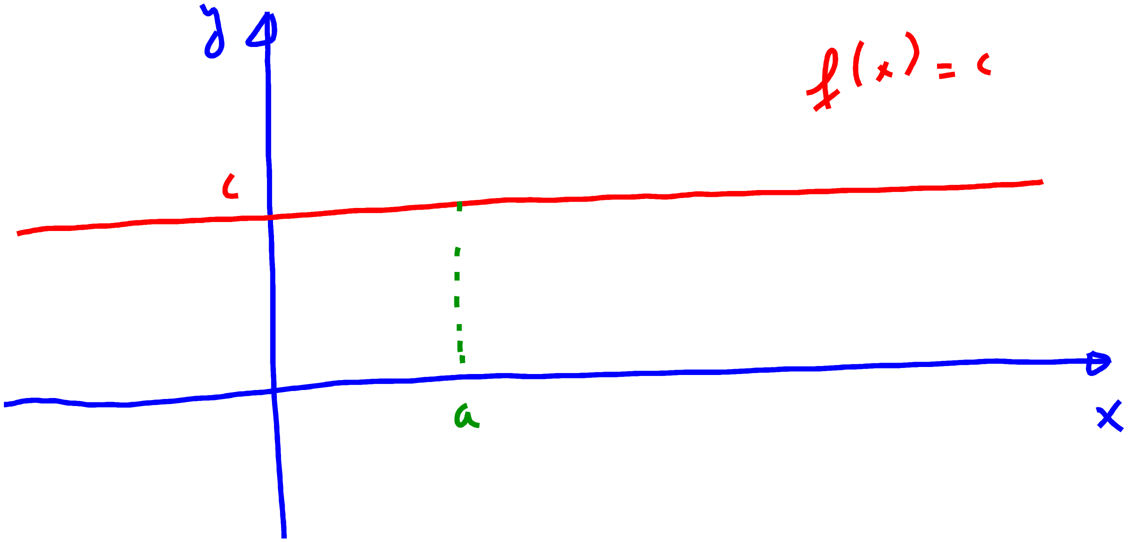
educated guess

what if we want to find limit as x approaches any arbitrary #.

Find limits analytically.

2 basic limits:

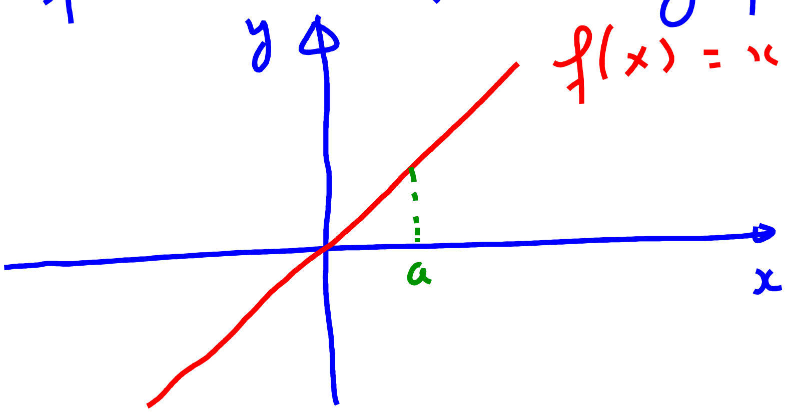
(1) c: constant. $f(x) = c$ for all x.



$$\lim_{x \to a} f(x) = c$$

In short, $\boxed{\lim_{x \to a} c = c}$

② $f(x) = x$. (Identity function)



$$\lim_{x \to a} f(x) = a$$

In short, $\boxed{\lim_{x \to a} x = a}$

3 Limit Laws

$\lim_{x \to a} f(x) = L$; $\lim_{x \to a} g(x) = M$

(L, M: finite #)

(a) <u>Sum Law</u>: $\lim_{x \to a} [f(x) + g(x)] = L + M$

or $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

(b) <u>Difference Law</u>: $\lim_{x \to a} [f(x) - g(x)] =$

$\lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(c) <u>Product Law</u>: $\lim_{x \to a} [f(x) g(x)] = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$.

(d) <u>Quotient Law</u>: $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$; (provided that $\lim_{x \to a} g(x) \neq 0$)

(e) Constant Multiple Law:

$$\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$$

(f) Root Law: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

(g) Power Law: $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

E.g. $\lim_{x \to -2} x^3 = \left(\lim_{x \to -2} x\right)^3 = (-2)^3 = -8$

Power law

$$\lim_{x\to 0}\left(4x^2-2x+3\right) = \lim_{x\to 0}(4x^2) - \lim_{x\to 0}(2x) + \lim_{x\to 0} 3$$

Sum / Difference law

$$= 4\boxed{\lim_{x\to 0} x^2} - 2\lim_{x\to 0} x + \lim_{x\to 0} 3$$

constant multiple law

$$= 4\cdot 0 - 2\cdot 0 + 3 = 3$$

$$\lim_{x\to -2}\sqrt{x^2-6x+3} = \sqrt{19}$$

<u>Laws:</u> Sum / Difference, constant multiple, Power, root, basic limits.

Bottom line:

When you try to find $\lim_{x \to a} f(x)$: you plug the # a into the function $f(x)$ and you get out a finite #, that # is the answer; i.e., it is the limit.

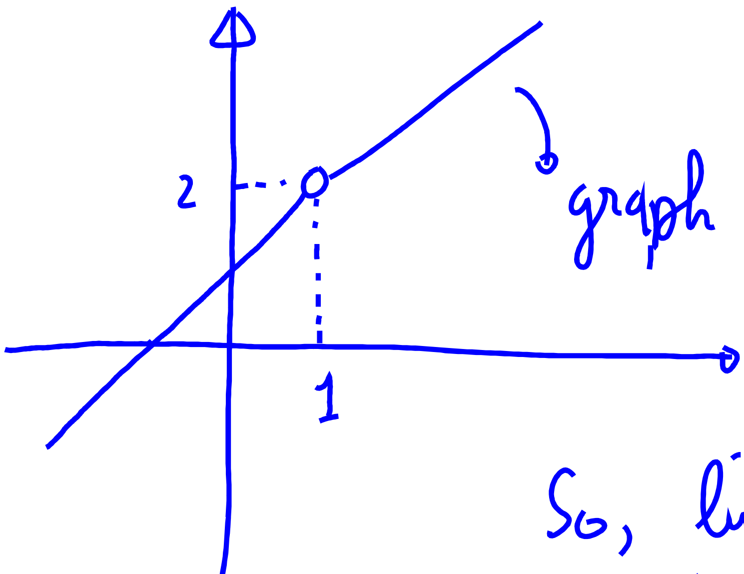
But for many functions, we will not get out a finite #, this does NOT mean the limit DNE.

E.g. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ Plug 1 into the function:

we get: $\frac{0}{0}$

$$\frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$$

if $x \neq 1$



graph of $\frac{x^2-1}{x-1}$

So, $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2 = \lim_{x \to 1} (x+1)$

This suggests the following technique to find limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where we get $\frac{0}{0}$ if we plug in a to $\frac{f(x)}{g(x)}$.

1. Factor numerator and denominator completely
2. Cancel all the common factors
3. Plug a into the simplified function.

E.g. 1 $\boxed{\lim_{x \to -3} \frac{x^2+4x+3}{x^2-9}}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to -3} \frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-3)}$$

$$= \boxed{\lim_{x \to -3} \frac{x+1}{x-3}}$$

$$= \frac{-2}{-6} = \boxed{\frac{1}{3}}$$

Laws: quotient, sum, difference, basic limits.

2 $$\lim_{h \to 0} \frac{(1+h)^2-1}{h} \quad \frac{0}{0} = \lim_{h \to 0} \frac{\cancel{1}+2h+h^2 \cancel{-1}}{h} = \lim_{h \to 0} \frac{2h+h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} (2+h) = \boxed{2}$$

Some variations of this technique

* $\frac{0}{0}$ limits that involve radicals

E.g. $\lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x-5} \quad \left(\frac{0}{0}\right)$

Multiply by the conjugate of the numerator:

$$= \lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x-5} \cdot \frac{\sqrt{x-1} + 2}{\sqrt{x-1} + 2}$$

$$= \lim_{x \to 5} \frac{x - 1 - 4}{(x-5)\left(\sqrt{x-1} + 2\right)} = \lim_{x \to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}\left(\sqrt{x-1} + 2\right)}$$

$$\lim_{x \to 5} \frac{1}{\sqrt{x-1} + 2}$$

$$= \boxed{\frac{1}{4}}$$

② $$\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = \lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} \cdot \frac{\sqrt{t}+3}{\sqrt{t}+3}$$

$$= \lim_{t\to 9} \frac{\cancel{(t-9)}\cdot(\sqrt{t}+3)}{\cancel{t-9}} = \lim_{t\to 9} (\sqrt{t}+3) = \boxed{6}.$$

* limits that involve complex fractions

E.g. ① $$\lim_{h\to 0} \frac{\frac{1\cdot 5}{(5+h)\cdot 5} - \frac{1\cdot(5+h)}{5\cdot(5+h)}}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{\frac{5}{(5+h)\cdot 5} - \frac{5+h}{(5+h)\cdot 5}}{h} = \lim_{h\to 0} \frac{\frac{\boxed{5-(5+h)}}{(5+h)\cdot 5}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{5}-\cancel{5}-h}{(5+h)\cdot 5}}{h} = \lim_{h\to 0} \frac{\boxed{\frac{-h}{(5+h)5}}}{\boxed{h}} = \lim_{h\to 0} \frac{-\cancel{h}}{(5+h)5} \cdot \frac{1}{\cancel{h}}$$

$$\lim_{h\to 0} \frac{-1}{(5+h)\cdot 5} = \boxed{\frac{-1}{25}}.$$

One-sided limits

$$f(x) = \begin{cases} \boxed{-x-2} & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ x^3 & \text{if } x > -1 \end{cases}$$

(piecewise-function)

$$\underbrace{\lim_{x \to -1^-} f(x)}_{} \;; \quad \underbrace{\lim_{x \to -1^+} f(x)}_{} \;; \quad \underbrace{\lim_{x \to -1} f(x)}_{} \;; \quad f(-1)$$

$$\lim_{x \to -1} (-x-2) = -1 \;; \quad \lim_{x \to -1} (x^3) = -1$$

$$\lim_{x \to -1} f(x) = -1$$

$$f(-1) = 2$$

* limits of the form $\frac{K}{0}$ where $K \neq 0$

E.g. $\lim\limits_{x \to 1} \frac{x+2}{(x-1)^2} = \infty$

$\frac{3}{0}$

$\lim\limits_{x \to 0} \frac{1}{x}$ DNE.

$\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$

$\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$

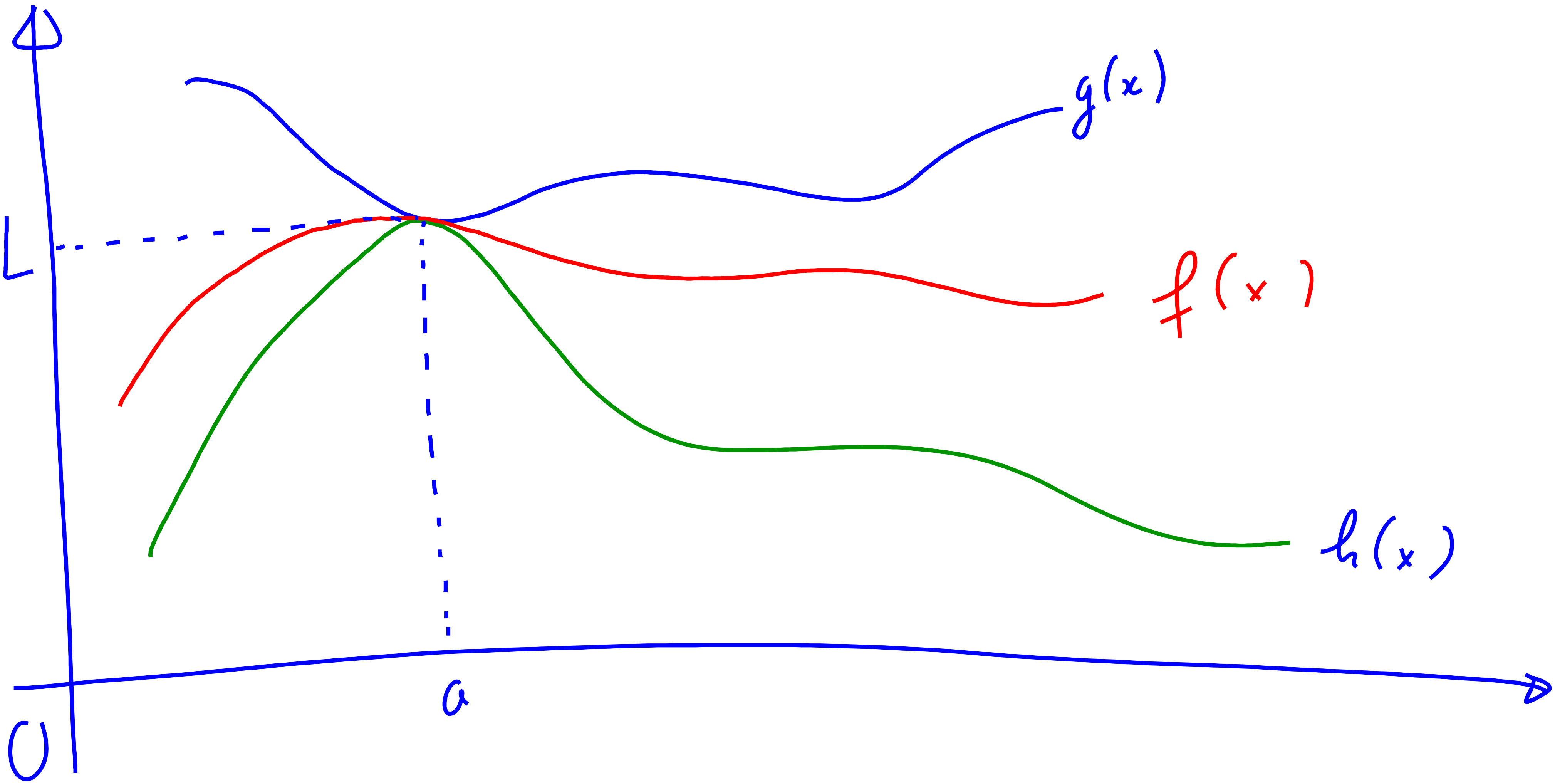
* limits that use the important limit $\boxed{\lim\limits_{x \to 0} \frac{\sin x}{x} = 1}$.

E.g. $\lim\limits_{x \to 0} \frac{\sin(2x)}{5x}$ $\left(\frac{0}{0}\right)$

$$\lim_{x \to 0} \frac{\sin(2x)}{5x} \cdot \frac{2x}{2x}$$

$$\lim_{x \to 0} \frac{\sin(2x)}{2x} \cdot \frac{2x}{5x} = \frac{2}{5}$$

E.g. $\lim\limits_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right)$ ⟵ Squeeze Theorem



g(x)
f(x)
h(x)
L
a
0

Squeeze Theorem:

If $h(x) \le f(x) \le g(x)$ for all x in an interval around the # a and

$$\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = L.$$

Then:

$$\lim_{x \to a} f(x) = L.$$

$$\lim_{x \to 0} \boxed{x^2 \cdot \sin\left(\frac{1}{x}\right)}$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2 \cdot \sin\left(\frac{1}{x}\right) \leq x^2$$

as $x \to 0$

$$0$$

So $\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$

by Squeeze Theorem.