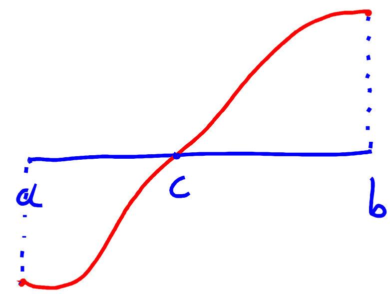
# Rolle's Theorem and Mean Value Theorem

## Existence Theorems.

We have seen I. V. T.

$f$ : continuous on $[a,b]$

$f(a)$ and $f(b)$ have different signs




$\Longrightarrow$ Conclusion: there exists $c$ in $(a,b)$ s.t. $f(c)=0$.

# Rolle's Theorem

Conditions (Requirements / Hypothesis)

$f$: function on $[a,b]$ ✓

$f$: continuous on $[a,b]$ ✓

$f$: differentiable on $(a,b)$ ✓

$f(a) = f(b)$ ✓

Hypothesis.

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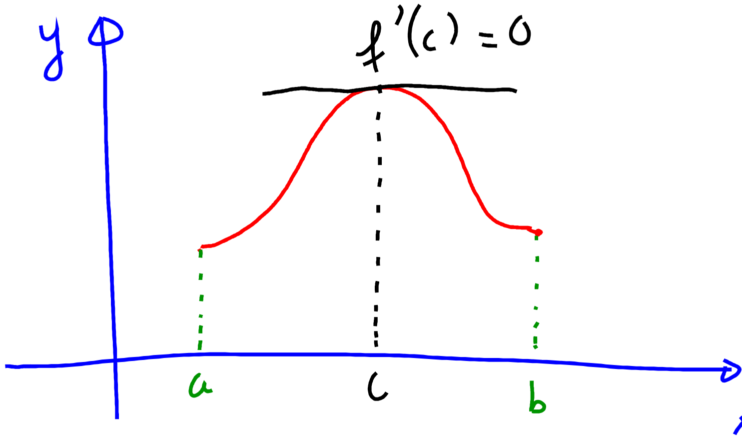
$\Longrightarrow$ <u>Conclusion</u>: there exists a number $c$ in $(a,b)$

such that $f'(c) = 0$




Picture of Rolle's Theorem.

$f(-2)=0$

$f(2)=0$

$f'(x) = 3x^2 - 4$

E.g. $f(x) = x^3 - 4x$ on $[-2, 2]$.

1. Verify that $f$ satisfies all the conditions of Rolle's Theorem. ✓

2. Find all values of $c$ for which $f'(c) = 0$ in $(-2, 2)$.

$3x^2 - 4 = 0$

$x^2 = \frac{4}{3}$ ; $x = \boxed{\pm \frac{2}{\sqrt{3}}}$ values of c

E.g. $x^3 + x - 1 = 0$.

Q: Prove that this equation has exactly one solution.

$f(-1) = -3 < 0$

$f(2) = 9 > 0$

$f$ is continuous on $[-1, 2]$

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By the IVT, there exists $c$ in $(-1, 2)$ s.t.
$$f(c) = 0$$

Suppose, by way of contradiction, there were 2 x-values $c$ and $d$ s.t. $f(c) = f(d) = 0$

Then Rolle's Theorem implies that there is a point $\alpha$ in $(c, d)$ s.t. $f'(\alpha) = 0$.

But $f'(x) = 3x^2 + 1 > 0$. This is a contradiction. This shows that $f$ could only have exactly one zero.

# Mean Value Theorem

$f$ : function on $[a,b]$

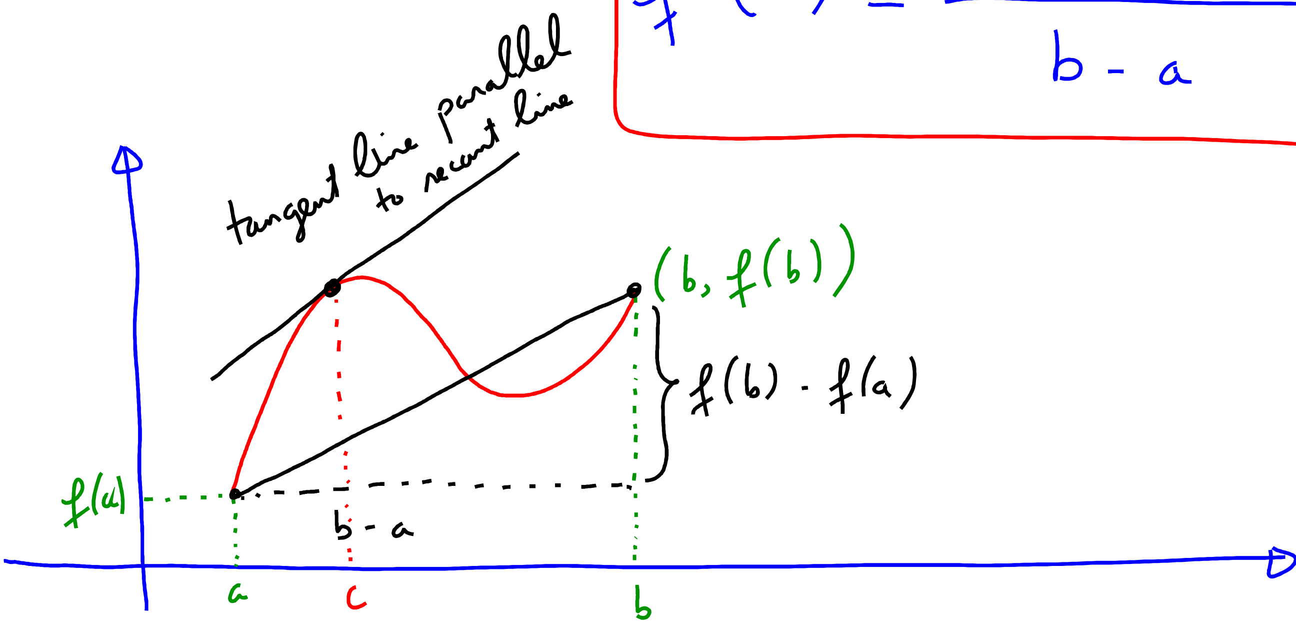
$f$ : continuous on $[a,b]$

$f$ : differentiable on $(a,b)$

hypothesis

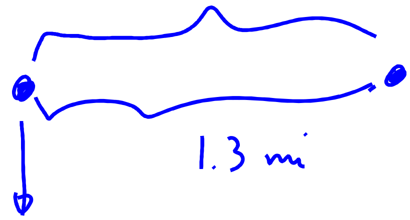
$\Longrightarrow$ Conclusion: there exists a number $c$ in $[a,b]$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$



* Patrolled by aircraft. Speed limit is 75 mi/h.

60 (s) : 1.3 mi.




there exists $c$ s.t.

$$\underbrace{f'(c)}_{\text{Instantaneous speed}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = 78 \text{ m/h}.$$

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E.g. HW #2. $f$ : cont. on $[0,8]$ ; ($-4$ is outside $[0,8]$)

$f'(x) = \dfrac{11}{(x+4)^2} \to$ diff. on $[0,8]$

$$f'(c) = \frac{f(8) - f(0)}{8 - 0} = \frac{11}{48}$$

$$\frac{11}{(c+4)^2} = \frac{11}{48} \; ; \; (c+4)^2 = 48 \; ; \; c = -4 \pm \sqrt{48}$$

$c = -4 + \sqrt{48}$ in $[0,8]$