Derivatives and Rates of Change

Recall: Power Rule $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

Sum/Difference $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$

Product: $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = \frac{d}{dx}\left(f(x)\right) \cdot g(x) + \frac{d}{dx}\left(g(x)\right) \cdot f(x)$

Quotient: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$.

Constant: $\frac{d}{dx}\left[c\,f(x)\right] = c\,\frac{d}{dx}\left[f(x)\right]$.

Recall: $y = f(x)$.

Then $\frac{dy}{dx} = f'(x) =$ instantaneous rate of change of the function f at the point x.

In particular, if $y = s(t)$: position function of an object at time t.

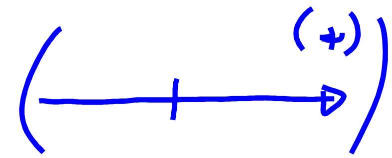
then :

$v(t) = s'(t)$ = velocity of object at time t.

$a(t) = v'(t) = s''(t)$ = acceleration of object at time t.

$|v(t)|$ = speed of object at time t.

<u>Eg</u>. The position of a particle moving along an axis



is given by : $s(t) = t^3 - 9t^2 + 24t + 4$; $t \geq 0$.

(a) At what time is the object at rest ?

(b) During what time interval is the particle moving from left to right $\xrightarrow{(+)}$? moving from right to left ?

(c) During what time interval is the particle speeding up ? slowing down ?

(d) Find distance traveled in the first 6 seconds?

(a) At rest: $\underbrace{v(t)}_{s'(t)} = 0$

(b) Moving in (+) direction: $v(t) > 0$

Moving in (−) direction: $v(t) < 0$.

(c) Speed up: velocity and acceleration act in the same direction

either $v(t) > 0$ and $a(t) > 0$

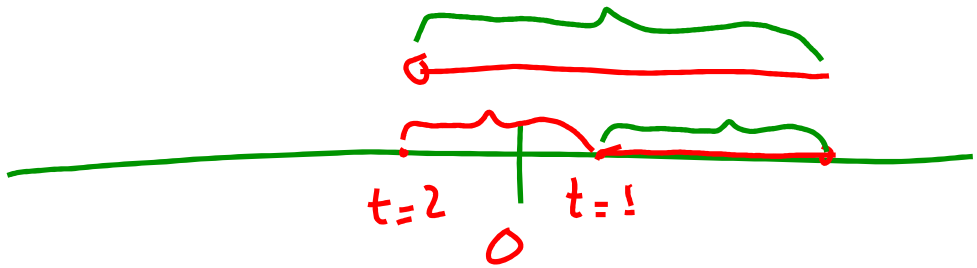
or $v(t) < 0$ and $a(t) < 0$

Slow down: they act in opposite direction

either $v(t) > 0$ and $a(t) < 0$

or $v(t) < 0$ and $a(t) > 0$

d



$$s(t) = t^3 - 9t^2 + 24t + 4 \; ; \; t \geq 0$$

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

$$a(t) = v'(t) = 6t - 18.$$

a <u>At rest</u>: $v(t) = 0$; $3t^2 - 18t + 24 = 0$

$$3(t^2 - 6t + 8) = 0$$

$$3(t-2)\cdot(t-4) = 0$$

$$\boxed{t = 2} \; ; \; \boxed{t = 4}$$

The time at which the particle is at rest are $t = 2$ (s) and $t = 4$ (s).

(c)

Speed up
- $v(t) > 0$; $a(t) > 0$
- $v(t) < 0$; $a(t) < 0$

($a(t) = 0$ when
$6t - 18 = 0$
$t = 3$

$\boxed{(2,3) \cup (4, \infty)}$

t	0		2		3		4	
$v(t)$		+	0	−		−	0	+
$a(t)$		−		−	0	+		+
		Slow down		Speed up		Slow down		Speed up

Slow down
- $v(t) > 0$; $a(t) < 0$
- $v(t) < 0$; $a(t) > 0$

on $\boxed{(0,2) \cup (3,4)}$

(b) Move to the right: $(0,2) \cup (4,\infty)$ $[v(t) > 0]$

Move to the left: $(2,4)$ $[v(t) < 0]$

(d) Distance traveled in 6 seconds.

$\boxed{[0,2] \; \longrightarrow}$; $[2,4] \; \longleftarrow$; $[4,6] \; \longrightarrow$

$|s(2) - s(0)|$; $|s(4) - s(2)|$; $|s(6) - s(4)|$.

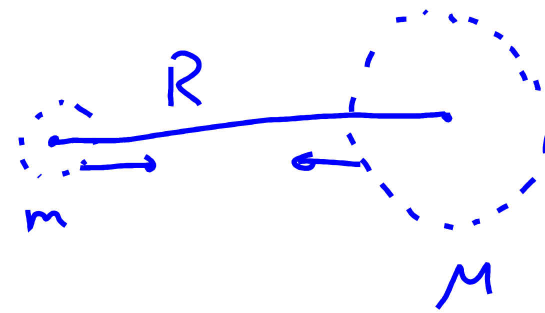
$s(t) = t^3 - 9t^2 + 24t + 4$

$s(2) - s(0) = (2)^3 - 9(2)^2 + 24(2) + 4 - 4 = \boxed{20}$

$|s(4) - s(2)| = \boxed{4}$; $|s(6) - s(4)| = \boxed{\ldots}$

E.g. Newton's Law of Gravitation

$$F = \frac{G m M}{R^2}$$



* Calculate $\dfrac{dF}{dR} = \, ?$

$$\frac{d}{dR}\left(\frac{G m M}{R^2}\right) = G m M \cdot \frac{d}{dR}\left(\frac{1}{R^2}\right)$$

$$= G m M \cdot \frac{d}{dR}\left(R^{-2}\right)$$

$$= G m M \cdot (-2) \cdot \frac{1}{R^3}$$

$$\boxed{\frac{dF}{dR} = \frac{-2 G m M}{R^3}}. \longrightarrow \text{R.o.C. of } F \text{ w.r.t. distance } R.$$

Suppose it is known that the earth attracts an object with force that decreases at a rate of 2 N/km when R = 20000 km. How fast does this force change when R = 10000 km?

$$F'(20000) = -2$$

$$F'(10000) = ?$$

$$F'(R) = -\frac{2GmM}{R^3}$$

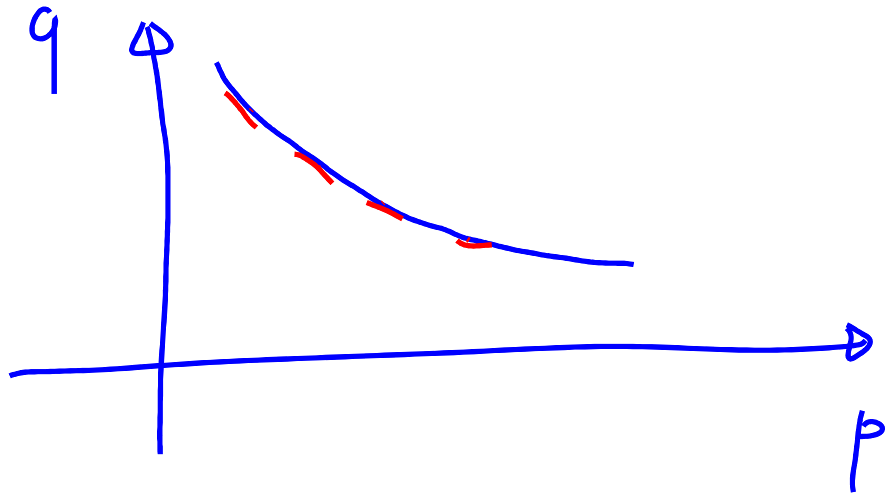
$$\underbrace{\frac{\cancel{-2}GmM}{(20000)^3}}_{F'(20000)} = \cancel{-2}.$$

$$\boxed{GmM = (20000)^3}$$

$$\frac{-2GmM}{(10000)^3} = \frac{-2\cdot(20000)^3}{(10000)^3} = -16 \text{ N/km}.$$

E.g. Price Elasticity of Demand

* Quantity demanded q is a function of price p.
  often, $q = q(p)$ is a decreasing function



$E > 1$: price elastic.
Increase in price $\rightarrow$ decrease in revenue

$E < 1$ price inelastic.
increase in price $\rightarrow$ increase in revenue

* Elasticity of demand :

$$\boxed{E = \left| \frac{p}{q(p)} \cdot \frac{dq}{dp} \right|}.$$

$E = 1 \rightarrow$ optimal price.

*

$\underline{\text{Revenue}} = p \cdot q(p)$

$x + y = x\left(1 + \frac{y}{x}\right).$

$R(p) = p \cdot q(p)$

Marginal revenue : $R'(p) = \boxed{p \cdot q'(p)} + q(p)$

$= q(p)\left[\frac{p \cdot q'(p)}{q(p)} + 1\right]$

$$\boxed{R'(p)} = \underbrace{\boxed{q(p)}}_{>0} \underbrace{\boxed{\left[1 + \underbrace{\boxed{\frac{p}{q(p)} \cdot \frac{dq}{dp}}}_{\text{negative}}\right]}}_{<0}$$

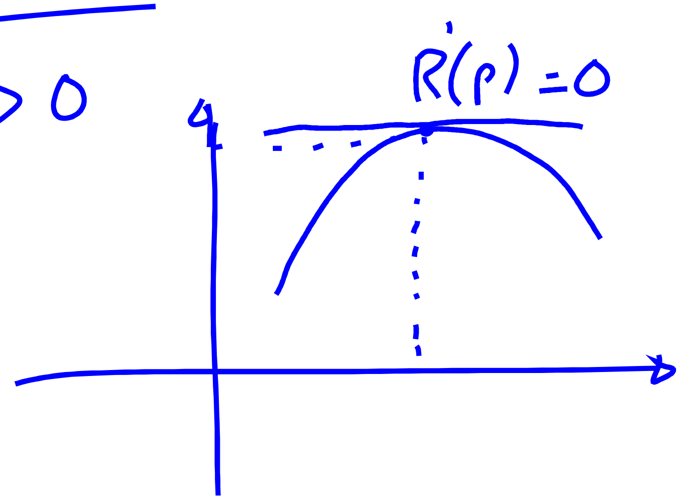
$$E > 1 \leftrightarrow \left|\frac{p}{q(p)} \cdot \frac{dq}{dp}\right| > 1$$

$$\leftrightarrow \frac{p}{q(p)} \cdot \frac{dq}{dp} < -1 \leftrightarrow R'(p) < 0$$

$$E < 1 \leftrightarrow \left| \frac{p}{q(p)} \cdot \frac{dq}{dp} \right| < 1 \leftrightarrow -1 < \frac{p}{q(p)} \cdot \frac{dq}{dp} < 0$$

$$\leftrightarrow R'(p) > 0$$

$$E = 1 \leftrightarrow R'(p) = 0$$



E.g. $q = 400 - p^2$.

(a) Find $\boxed{E(\$5)}$. What does this tell you?

(b) Find $E(\$15)$.

(c) When is $E = 1$?

$$\underline{E(p)} = \boxed{\left| \frac{p}{q(p)} \cdot \frac{dq}{dp} \right| = 1}$$

$$q = 400 - p^2; \quad \frac{dq}{dp} = -2p$$

$$\left| \frac{p}{400-p^2} \cdot (-2p) \right| = 1 \qquad \left| \frac{-2p^2}{400-p^2} \right|$$

$$\frac{2p^2}{400-p^2} = 1;$$

$$2p^2 = 400 - p^2$$

$$3p^2 = 400$$

$$p = \sqrt{\frac{400}{3}}$$

$$p = 5; \quad q(5) = 400 - 25 = 375$$

$$\frac{dq}{dp} = -2 \cdot 5 = -10; \quad E(5) = \left| \frac{5}{375} \cdot (-10) \right|$$

$$= 0.123 < 1$$

$\rightarrow$ inelastic.

$$p = 15; \quad q(15) = 400 - 225 = 175$$

$$\frac{dq}{dp} = -2 \cdot 15 = -30; \quad E(15) = \left| \frac{15}{175} \cdot (-30) \right| > 1$$

$\rightarrow$ elastic.