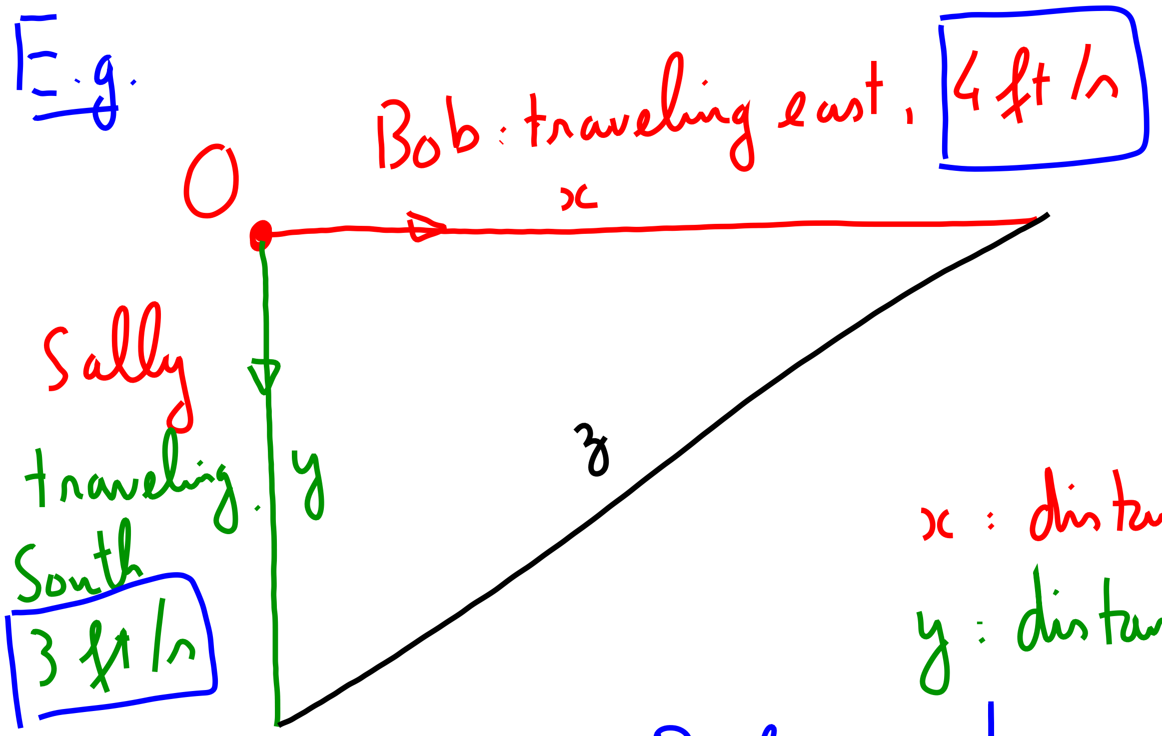
4.1. Related Rates

E.g.



Q: 10 s later

At what rate is the distance between Bob and Sally are changing?

x : distance from Bob to O

y : distance from Sally to O

Both x and y are changing with time.

z : distance between Bob and Sally. z is changing with time

Want: $\frac{dz}{dt}$, when $t = 10$ s. $\frac{dx}{dt} = 4$ ft/s; $\frac{dy}{dt} = 3$ ft/s.

⟶ Relation among the "changing" quantities x, y, z:

$$z^2 = x^2 + y^2.$$

⟶ take the derivative w.r.t. time t of both sides

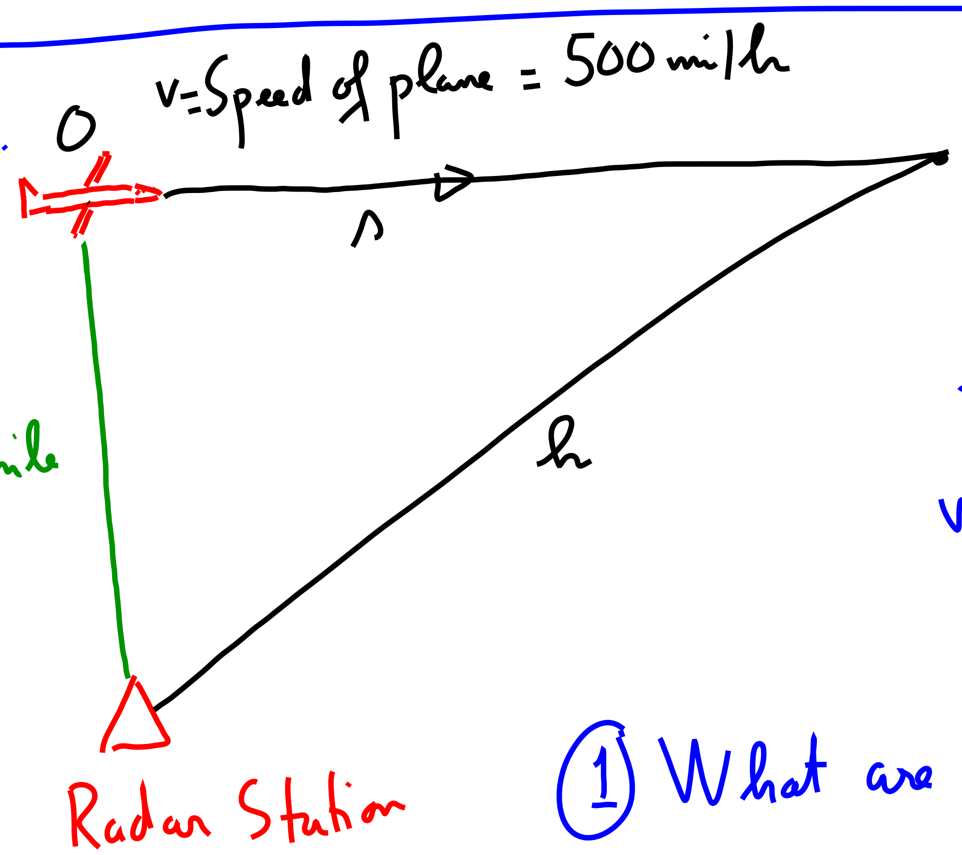
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \text{(Chain Rule)}$$

When $t = 10(s)$; $x = 40$ ft, $y = 30$ ft

$$z = \sqrt{30^2 + 40^2} = 50 \text{ ft}.$$

Solve for $\frac{dz}{dt} = 5$ ft/s.

E.g.



Q: Find the rate at which the distance from the plane to the station is increasing when it is $\boxed{2 \text{ mi}}$ away from the station?

1. What are "changing" quantities?
2. Name them
3. Relationship between them.
4. Relationship btwn their rates of change.

s: distance from plane to O.

h: distance from plane to station

Relationship btwn them: $\boxed{s^2 + 1 = h^2}$.

$s^2 + 1 = 4$

$s^2 = 3$;

$s = \sqrt{3}$

Know: $\frac{ds}{dt} = 500$

Want: $\frac{dh}{dt}$ when $\boxed{h = 2}$

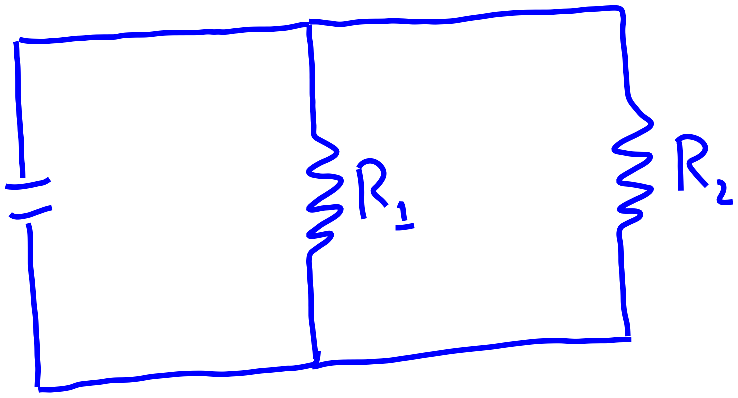
$\frac{d}{dt}$

$$2\boxed{s}\boxed{\frac{ds}{dt}} = 2\boxed{h}\boxed{\frac{dh}{dt}}$$

500 2 ?

$$2 \cdot \sqrt{3} \cdot 500 = 2 \cdot 2 \cdot \frac{dh}{dt}; \quad \frac{dh}{dt} = \frac{1000\sqrt{3}}{4} = 250\sqrt{3} \text{ mi/h}$$

E.g. Parallel



2 resistors with resistances R_1 and R_2.

R: total resistance.

R_1: increasing at a rate of $0.3\ \Omega/s$

R_2: ——— $0.2\ \Omega/s$.

How fast is R changing when $R_1 = 80\Omega$ and $R_2 = 100\ \Omega$?

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}}$$

Know: $\dfrac{dR_1}{dt} = 0.3\ \Omega/s$. $\dfrac{dR_2}{dt} = 0.2\ \Omega/s$

Want: $\dfrac{dR}{dt} = ?$

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}} \xrightarrow{\frac{d}{dt}} \frac{d}{dt}\left(\frac{1}{R}\right) = \frac{d}{dt}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

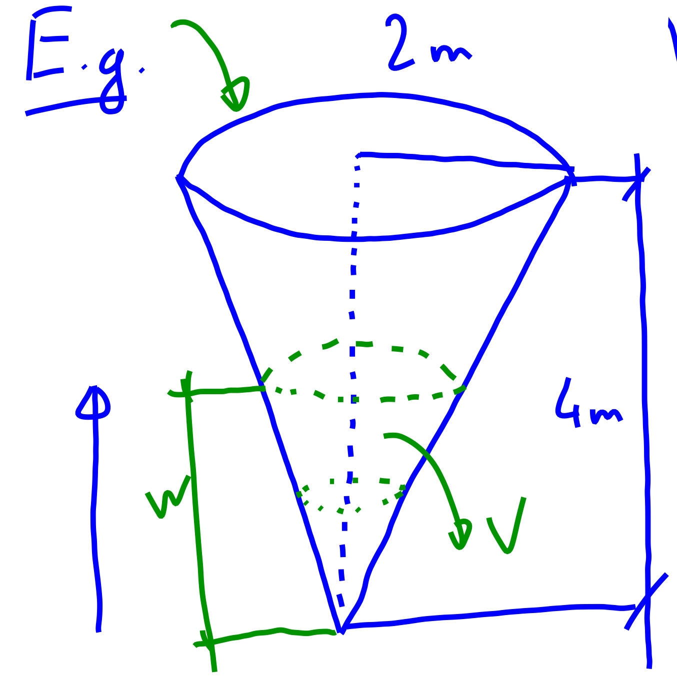
$$-\frac{1}{R^2}\cdot\frac{dR}{dt} = -\frac{1}{R_1^2}\cdot\boxed{\frac{dR_1}{dt}} - \frac{1}{R_2^2}\boxed{\frac{dR_2}{dt}}$$

$\frac{400}{9}$ 80 0.3 100 0.2

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$$

$$R = \frac{400}{9}$$

E.g.



Water tank. Inverted circular cone

Base radius: 2 m

Height: 4 m

Water is pumped into tank at a rate of $2\ m^3/min$.

Find rate at which the water level is rising when water reaches a depth of 3m.

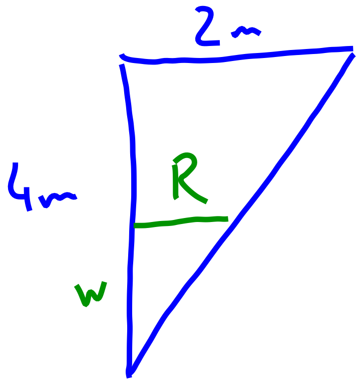
1. Quantities that are changing
2. Name them
3. Relation among these
4. Differentiate

w: height of water (water level)

V: Volume of water. $\frac{dV}{dt} = 2\ m^3/min.$

Want: $\frac{dw}{dt}$ when $w = 3$

Relation:



$$V = \frac{\pi}{3} \cdot w \cdot R^2$$

Similar triangle: $\frac{R}{w} = \frac{2}{4} = \frac{1}{2} \Rightarrow R = \frac{w}{2}$.

$$V = \frac{\pi}{3} \cdot w \cdot \left(\frac{w}{2}\right)^2 = \frac{\pi}{12} w^3$$

$$\boxed{\frac{dV}{dt}} = \frac{\pi}{12} \cdot 3 w^2 \cdot \frac{dw}{dt}$$

2   3   ?