Integration by Substitution

$$\int \sin(x)\,dx = -\cos x + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \;;\; n \neq -1$$

$$\int \sin(u)\,du = -\cos(u) + C$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \;;\; n \neq -1$$

Substitution method (u-substitution)

It works if the integral has the form:

$$\boxed{\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du}} \to \boxed{\int f(u)\,du}$$

$g(x) = u$

$du = g'(x)\,dx$

E.g. $\displaystyle\int \underbrace{\boxed{\sin}}_{f}(\underbrace{x^3}_{g(x)}) \cdot \underbrace{3x^2}_{g'(x)}\, dx$

Let $u = x^3$. Then $du = 3x^2\, dx$

$$\int \sin(u)\, du = -\cos(u) + C$$
$$= \boxed{-\cos(x^3) + C}$$

<u>Check:</u> $\left(-\cos(x^3)\right)' = \sin(x^3) \cdot 3x^2$

E.g. $\displaystyle\int \sin(x^3) \cdot 9x^2\, dx = \int \sin(x^3) \cdot 3 \cdot 3x^2\, dx$

$$3\int \sin(\underbrace{x^3}_{u}) \cdot \underbrace{3x^2\,dx}_{du} = 3\int \sin(u)\,du$$

$$= -3\cos(u) + C$$

$$= -3\cos(x^3) + C.$$

Eg. $\frac{1}{3}\int \sin(x^3)\,3x^2\,dx = \frac{1}{3}\int \sin(u)\,du$

$$= -\frac{1}{3}\cos(u) + C$$

$$= \boxed{-\frac{1}{3}\cos(x^3) + C}$$

More examples of Integration by Substitution.

E.g. $\frac{1}{3}\int \frac{3x^2}{x^3+6}\,dx$

Let $u = x^3 + 6$

$du = 3x^2\,dx$

The original integral becomes

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$$

$$= \boxed{\frac{1}{3}\ln|x^3+6| + C}$$

E.x.

① $\int \cos^2(x)\sin(x)\,dx$

② $\int z\sqrt{z^2-5}\,dz$

③ $\int \frac{\sin(t)}{\cos^3(t)}\,dt$

Ans:

① $u = \cos(x)$. ans: $-\frac{\cos^3(x)}{3} + C$

② $u = z^2 - 5$. ans: $\frac{(z^2-5)^{3/2}}{3} + C$

③ $u = \cos(t)$. ans: $\frac{1}{2\cos^2(t)} + C$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$u = \cos x$

Ans: $-\ln|\cos(x)| + C$ or $\ln|\sec(x)| + C$.

$$\int \sec(x)\,dx = \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\,dx$$

$$= \int \frac{\overbrace{\sec^2(x) + \sec(x)\tan(x)\,dx}^{du}}{\underbrace{\sec(x) + \tan(x)}_{u}} = \ln|\sec(x) + \tan(x)| + C$$

$u = \sec(x) + \tan(x)$. $du = (\sec(x)\tan(x) + \sec^2(x))\,dx$

u- substitution for definite integrals

$$\frac{1}{6}\int_0^1 \left(\boxed{1+2x^3}\right)^5 \boxed{6x^2\,dx}$$

$\underbrace{6x^2\,dx}_{du}$

Let $u = 1 + 2x^3$. $\quad du = 6x^2\,dx$

$$\frac{1}{6}\int u^5\,du$$

<u>1st way:</u> find the new limits of integration for u.

$x = 0$; $u = 1$. $\quad x = 1$; $u = 3$

$$\frac{1}{6}\int_1^3 u^5\,du = \left.\frac{u^6}{36}\right|_1^3 = \frac{(3)^6}{36} - \frac{1}{36}.$$

2^{nd}. Find antiderivative in terms of x and use limits for x.

$$\frac{1}{6}\int u^5\,du = \frac{u^6}{36} = \frac{(1+2x^3)^6}{36}$$

$$\int_0^1 (1+2x^3)^5 x^2\,dx \underset{\text{FTC}}{=} \left.\frac{(1+2x^3)^6}{36}\right|_0^1 = \frac{3^6}{36} - \frac{1}{36}.$$