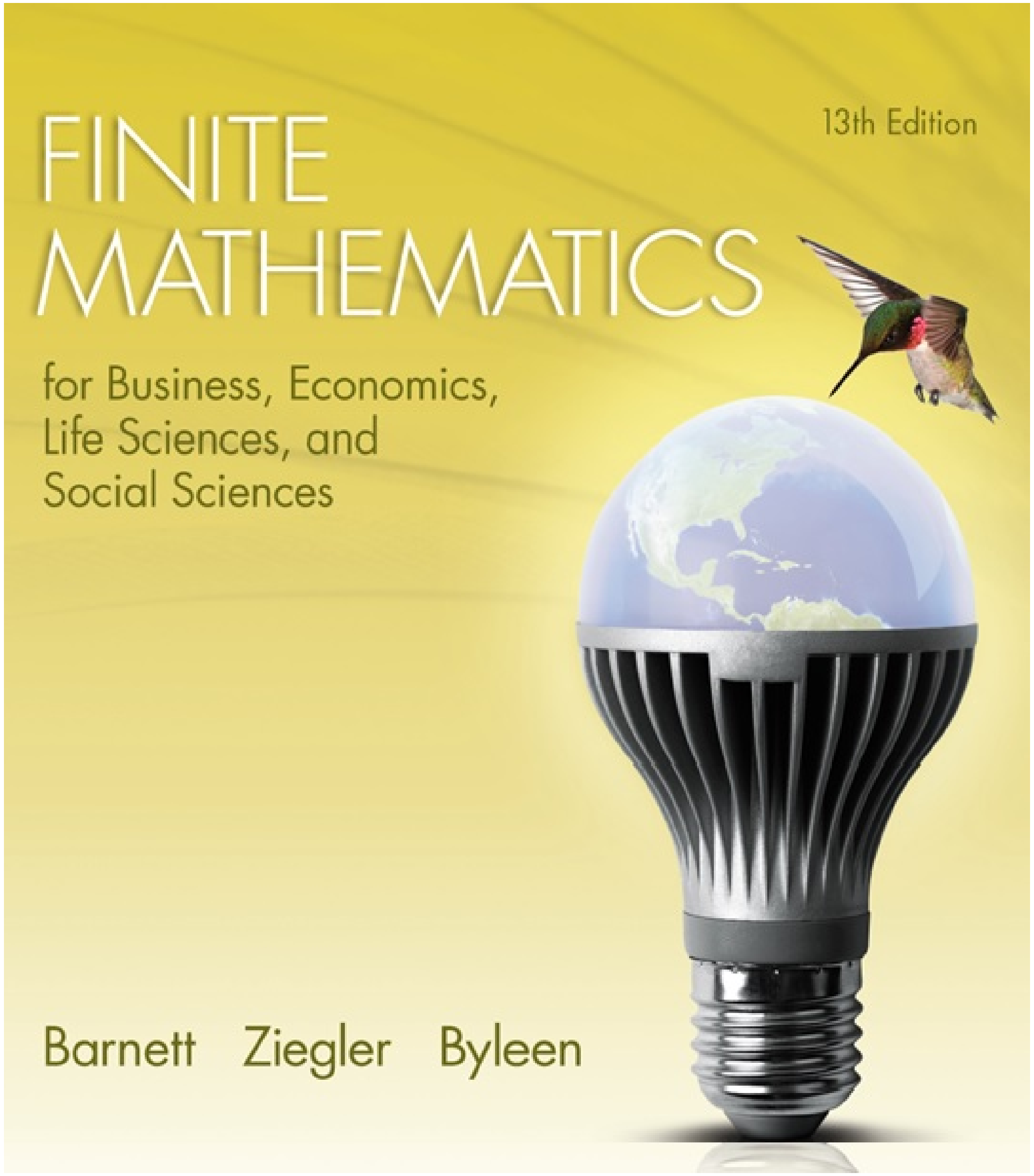



# Chapter 11

## Data Descriptions and Probability Distributions

## Section 4

## Bernoulli Trials and Binomial Distribution

# Learning Objectives for Section 11.4

## Bernoulli Trials and Binomial Distributions

- The student will be able to construct a Bernoulli experiment or trial.
- The student will be able to use the binomial formula.
- The student will be able to construct a binomial distribution.
- The student will be able to solve applications involving Bernoulli trials and binomial distributions.

# Bernoulli Trials

Boy? Girl? Heads? Tails? Win? Lose? Do any of these sound familiar? When there is the possibility of only two outcomes occurring during any single event, it is called a **Bernoulli Trial**. Jakob Bernoulli, a profound mathematician of the late 1600s, from a family of mathematicians, spent 20 years of his life studying probability. During this study he arrived at an equation that calculates probability in a Bernoulli Trial. His proofs are published in his 1713 book *Ars Conjectandi* (Art of Conjecturing).

# Jacob Bernoulli
# 1654-1705

Hofmann sums up Jacob Bernoulli's contributions as follows:

- *"Bernoulli greatly advanced algebra, the infinitesimal calculus, the calculus of variations, mechanics, the theory of series, and the theory of probability. He was self-willed, obstinate, aggressive, vindictive, beset by feelings of inferiority, and yet firmly convinced of his own abilities. With these characteristics, he necessarily had to collide with his similarly disposed brother. He nevertheless exerted the most lasting influence on the latter."*

- *"Bernoulli was one of the most significant promoters of the formal methods of higher analysis. Astuteness and elegance are seldom found in his method of presentation and expression, but there is a maximum of integrity."*

# What Constitutes a Bernoulli Trial?

- To be considered a **Bernoulli trial**, an experiment must meet each of three criteria:
  - There must be **only 2 possible outcomes**, such as black or red, sweet or sour. One of these outcomes is called a **success**, and the other a **failure**. Successes and Failures are denoted as S and F, though the terms given do not mean one outcome is more desirable than the other.
  - Each outcome has a **fixed probability** of occurring; a success has the probability of $p$, and a failure has the probability of $1 – p$.
  - Each experiment is completely **independent** of all others.

# Examples of Bernoulli Trials

- Flipping a coin. In this context, obverse ("heads") denotes success and reverse ("tails") denotes failure. A fair coin has the probability of success 0.5, by definition.
- Rolling a die, where for example we designate a six as "success" and everything else as a "failure".
- In conducting a political opinion poll, choosing a voter at random to ascertain whether that voter will vote "yes" in an upcoming referendum.
- Call the birth of a baby of one sex "success" and of the other sex "failure." (Take your pick.)

http://en.wikipedia.org/wiki/Bernoulli_trial

# Binomial Probability Solution

**Example:** A manager of a department store has determined that there is a probability of 0.30 that a particular customer will buy at least one product from his store. If three customers walk in a store, find the probability that exactly two of three customers will buy at least one product.

**Solution:** For each customer, there are two possible outcomes: "buy" (*b*) or "not buy" (*b*'). Each customer is independent of the others. If "buy" is a success, the probability of success is 0.30. There are three possible outcomes consisting of two *b* and one *b'*: *b b b'* (first two buy and third does not buy), *b b' b*, and *b' b b*.

# Binomial Probability (continued)

Since the trials are independent, we can use the probability rule for independence: $p(A \cap B \cap C) = p(A)*p(B)*p(C)$ .

For the outcome $b\ b\ b'$ , the probability is

$$\mathrm{p}(b\ b\ b') = p(b)\ p(b)\ p(b') = (0.30)(0.30)(0.70).$$

For the other two outcomes, the probability will be the same. The order in which the customers buy or not buy is not important.

We can use the formula for combinations to determine the number of ways two “buying” customers can be selected from a set of three customers: $C(3, 2) = 3$. For each of these three combinations, the probability is the same.

# Binomial Probability (continued)

Thus, we have the following formula to compute the probability that two out of three customers will buy at least one product :

$$C(3,2)\bullet 0.30^2 \cdot 0.70^1$$

This turns out to be 0.189.

Using the results of this problem, we can generalize the result. Suppose you have *n* customers and you wish to calculate the probability that *x* out of the *n* customers will buy at least one product. Let *p* represent the probability that a customer will buy a product. Then (1-*p*) is the probability that a given customer will not buy the product.

$$p(x) = C(n,x)\cdot p^x(1-p)^{n-x}$$

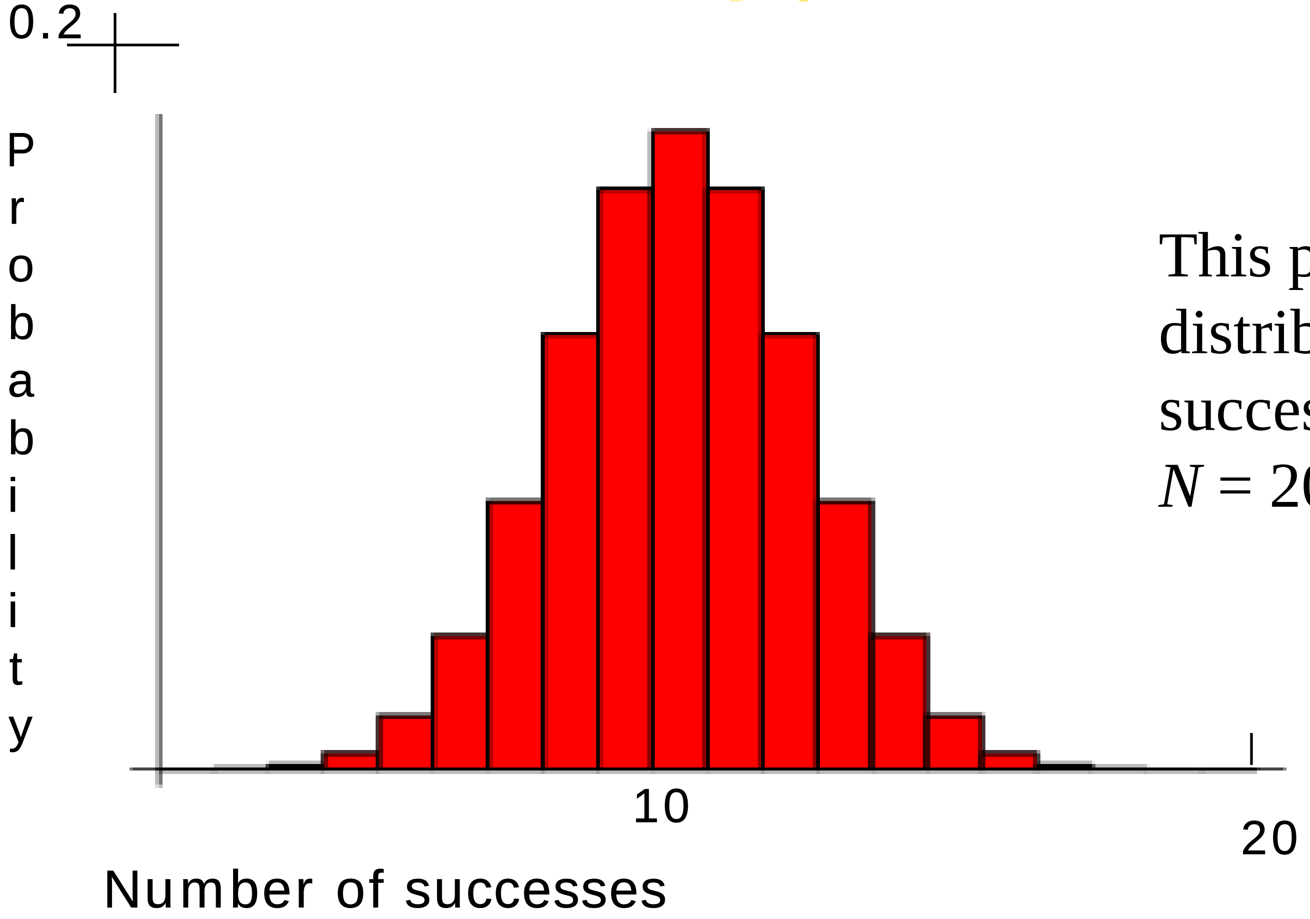
# Binomial Probability Formula

- The binomial distribution gives the discrete probability distribution of obtaining exactly $n$ successes out of $N$ Bernoulli trials (where the result of each Bernoulli trial is a success with probability $p$ and failure with probability 1-$p$). The binomial distribution is therefore given by

(1) $$\binom{N}{n} p^n q^{N-n}$$

(2) $$\frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

# Plot of Binomial Probabilities With $N = 20$, $p = 0.5$




This plot shows the distribution of $n$ successes out of $N = 20$ trials.

# Finding a Binomial Probability Formula

**Assumptions:**

1. $n$ identical trials
2. Two outcomes, success or failure, possible for each trial
3. Trials are independent
4. Probability of success $p$ remains constant on each trial

**Step 1:** Identify a success

**Step 2**: Determine $p$, the success probability

**Step 3**: Determine $n$, the number of trials

**Step 4:** The binomial probability formula for the number of successes $x$ is:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

# Example of Binomial Probability

Studies show that 60% of US families use physical aggression to resolve conflict. If 10 families are selected at random, find the probability that the number that use physical aggression to resolve conflict is:

- exactly 5
- between 5 and 7 , inclusive
- over 80 % of those surveyed
- fewer than nine

# Example (continued)

**Solution:**

- $p(x = 5) = \binom{10}{5} \cdot 0.6^{5}(1 - 0.6)^{(10-5)} = 0.201$

- $p(5 \le x \le 7) = p(5) + p(6) + p(7) =$

$$\binom{10}{5} 0.60^{5}(0.40)^{5} + \binom{10}{6}(0.6)^{6}(0.4)^{4} + \binom{10}{7}(0.6)^{7}(0.4)^{3}$$

$$= 0.20066 + 0.25082 + 0.1499 = 0.66647$$

# Example (continued)

- The probability that the number of families that use physical aggression is over 80% of those surveyed is $p$(9 or 10), since 10 families were surveyed.

$$P(9)+P(10)=\binom{10}{9}0.6^{9}0.4^{1}+\binom{10}{10}0.6^{10}$$

$$= 0.04031 + 0.00605 = 0.04636$$

# Example (continued)

- To compute the probability that $x$ is fewer than 9, we can just take the complement of the event that $x$ is 9 or 10, which we computed in the last example. So $p(x < 9) = 1 – 1\ \ p(x = 9 \text{ or } 10) =$

  $1 – 0.04636 = 0.95364.$

# Example

- Suppose 15% of major league baseball players are left-handed. In a sample of 12 major league baseball players, find the probability that
  - (a) none are left handed:
  - (b) at most six are left handed.
- **Solution:**
  - $C(12,0) \bullet 0.85^{12} = 0.1422$
  - Find probability of 0,1,2,3,4,5,6 and add:

    0.1422 + 0.30122 + 0.29236 + 0.17198 + 0.06828 + 0.01928 + 0.00397 = 0.99929.

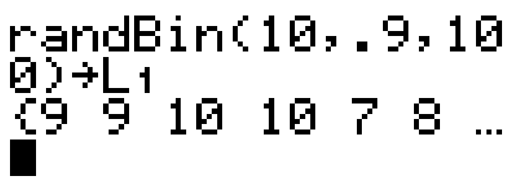
# Another Example

- A basketball player shoots 10 free throws. The probability of success on each shot is 0.90. Is this a binomial experiment? Why?
- **Answer:** Yes, it is a binomial experiment because there are two outcomes, success and failure, which are independent, and the probabilities remain constant.
- Use Excel or a graphing utility to compute the probabilities and draw the histogram of the results.

# Example (continued)

The graphing utility command shown below simulates 100 repetitions of the binomial experiment. The number of successes in each trial is stored in list $L_1$. From the second figure, we see that the empirical probability of tossing 9 free throws out of 10 is 41/100, which is close to the theoretical probability of

$C(10, 9) \times 0.9^9 \times 0.1 = 0.3874.$




# Mean of a Binomial Distribution

- To find the mean of a binomial distribution, multiply the number of trials *n* by the success probability of each trial:

$$\text{Mean} = np$$

- **Note:** This formula can only be used for the binomial distribution and not for probability distributions in general

# Example

- A large university has determined from past records that the probability that a student who registers for fall classes will have his or her schedule rejected (due to overfilled classrooms, clerical error, etc.) is 0.25. Find the mean number of rejected schedules in a sample of 20 students.
- **Answer:** The mean is $20 \times (0.25) = 5$. This means that, on the average, in a sample of size 20 you will have 5 rejections.

# Standard Deviation of the Binomial Distribution

The standard deviation of the binomial distribution is given by

$$\sigma = \sqrt{np(1-p)}$$

**Example:** Find the mean and standard deviation of the binomial distribution of *x*, the number of heads that appear when 100 coins are tossed.

**Solution:** Mean = $np = 100 \times 0.5 = 50$.

Standard deviation = $\sigma = \sqrt{np(1-p)}$ = 5.