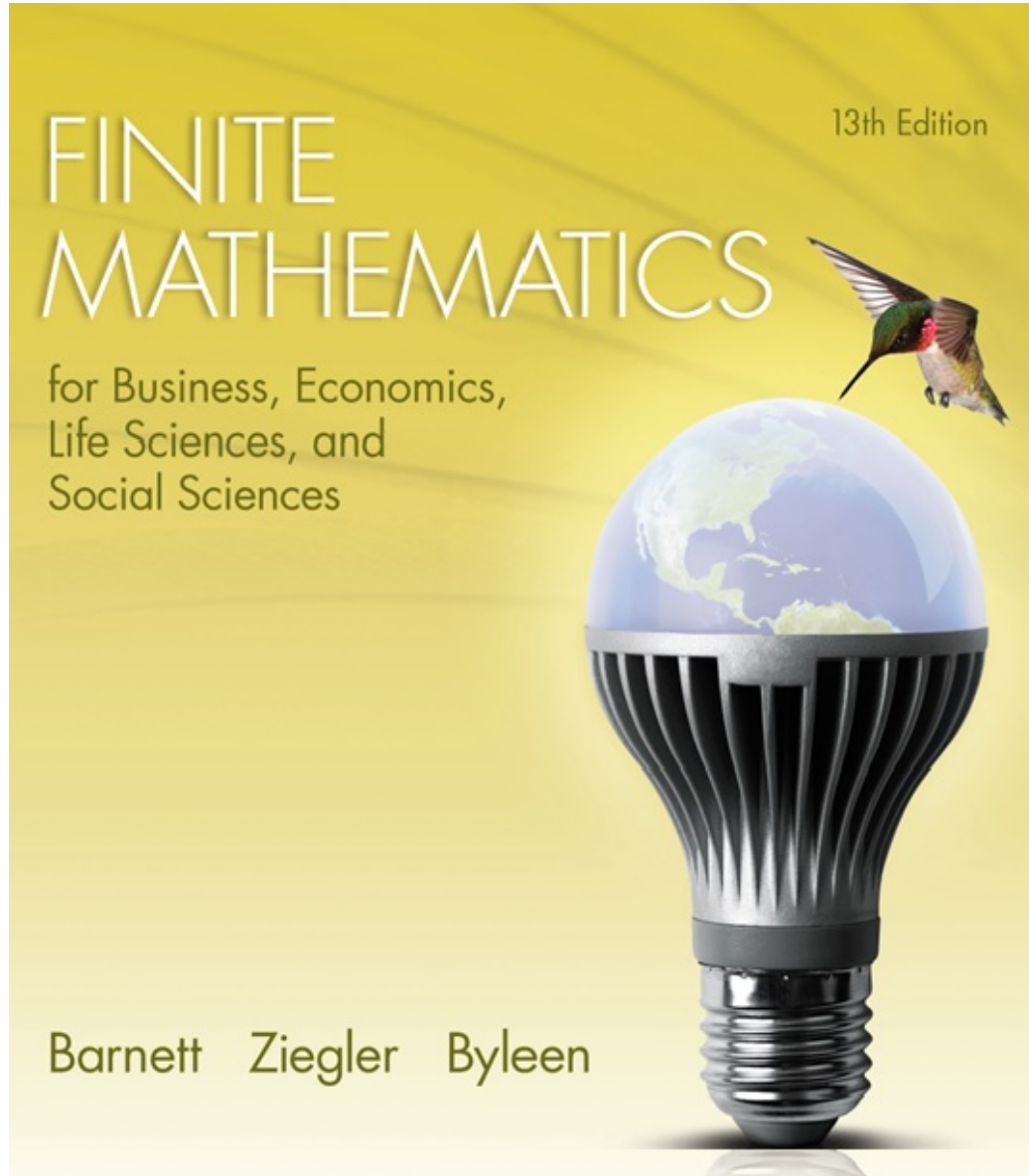




Chapter 3

Mathematics of Finance

Section 2

Compound and Continuous Compound Interest

Learning Objectives for Section 3.2



Compound and Continuous Compound Interest

- The student will be able to compute compound and continuous compound interest.
- The student will be able to compute the growth rate of a compound interest investment.
- The student will be able to compute the annual percentage yield of a compound interest investment.

Compound Interest



- Unlike simple interest, **compound interest** on an amount accumulates at a faster rate than simple interest. The basic idea is that after the first interest period, the amount of interest is added to the principal amount and then the interest is computed on this higher principal. The latest computed interest is then added to the increased principal and then interest is calculated again. This process is completed over a certain number of compounding periods. The result is a much faster growth of money than simple interest would yield.

Example



Suppose a principal of \$1.00 was invested in an account paying 6% annual interest compounded monthly. How much would be in the account after one year?

Solution

- **Solution:** Using the simple interest formula $A = P(1 + rt)$ we obtain:

- amount after one month $1 + \frac{0.06}{12}(1) = 1(1 + 0.005) = 1.005$

- after two months $1.005(1 + \frac{0.06}{12}) = 1.005(1.005) = 1.005^2$

- after three months $1.005^2 \left(1 + \frac{0.06}{12}\right) = 1.005^2(1.005) = 1.005^3$

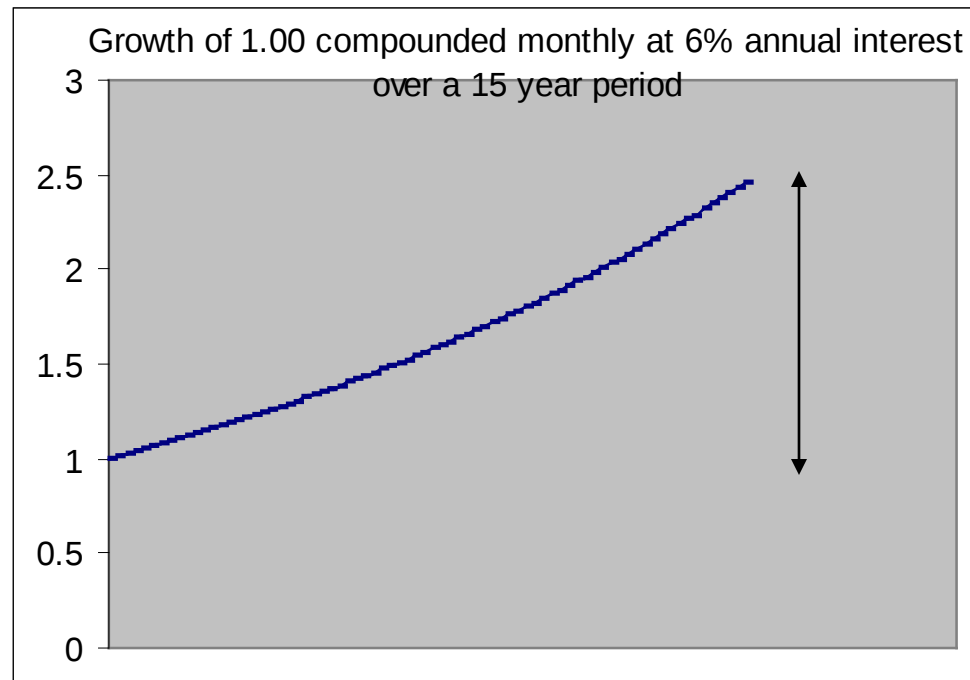
After 12 months, the amount is $1.005^{12} = 1.0616778$.

With simple interest, the amount after one year would be 1.06.

The difference becomes more noticeable after several years.

Graphical Illustration of Compound Interest

Growth of 1.00 compounded monthly at 6% annual interest over a 15 year period (Arrow indicates an increase in value of almost 2.5 times the original amount.)



General Formula

- In the previous example, the amount to which 1.00 will grow after n months compounded monthly at 6% annual interest is $\left(1 + \frac{0.06}{12}\right)^n = (1.005)^n$
- This formula can be generalized to $A = P \left(1 + \frac{r}{m}\right)^{mt}$

where A is the future amount, P is the principal, r is the interest rate as a decimal, m is the number of compounding periods in one year, and t is the total number of years. To simplify the formula,

$$A = P(1 + i)^n \quad i = \frac{r}{m}$$

$$n = mt$$

Compound Interest General Formula



$$A = P(1 + i)^n$$

where $i = r/m$

A = amount (future amount) at the end of n periods

P = principal (present value)

r = annual nominal rate

m = number of compounding periods per year

i = rate per compounding period

t = total number of compounding periods

Example



- Find the amount to which \$1500 will grow if compounded quarterly at 6.75% interest for 10 years.

Example

- Find the amount to which \$1500 will grow if compounded quarterly at 6.75% interest for 10 years.
- **Solution:** Use

$$A = P(1 + i)^n$$

$$A = 1500 \left(1 + \frac{0.0675}{4} \right)^{10(4)}$$

$$A = 2929.50$$

- Helpful hint: Be sure to do the arithmetic using the rules for order of operations.

Same Problem Using Simple Interest



Using the simple interest formula, the amount to which \$1500 will grow at an interest of 6.75% for 10 years is given by

$$\begin{aligned} A &= P (1 + rt) \\ &= 1500(1 + 0.0675(10)) = 2512.50 \end{aligned}$$

which is more than \$400 less than the amount earned using the compound interest formula.

Changing the number of compounding periods per year



To what amount will \$1500 grow if compounded *daily* at 6.75% interest for 10 years?

Changing the number of compounding periods per year

To what amount will \$1500 grow if compounded *daily* at 6.75% interest for 10 years?

$$A = 1500 \left(1 + \frac{0.0675}{365} \right)^{10(365)}$$

Solution: $= 2945.87$

This is about \$15.00 more than compounding \$1500 quarterly at 6.75% interest.

Since there are 365 days in year (leap years excluded), the number of compounding periods is now 365. We divide the annual rate of interest by 365. Notice, too, that the number of compounding periods in 10 years is $10(365) = 3650$.

Effect of Increasing the Number of Compounding Periods



- If the number of compounding periods per year is increased while the principal, annual rate of interest and total number of years remain the same, the future amount of money will increase slightly.

Computing the Inflation Rate



- Suppose a house that was worth \$68,000 in 1987 is worth \$104,000 in 2004. Assuming a constant rate of inflation from 1987 to 2004, what is the inflation rate?

Computing the Inflation Rate Solution

- Suppose a house that was worth \$68,000 in 1987 is worth \$104,000 in 2004. Assuming a constant rate of inflation from 1987 to 2004, what is the inflation rate?

- Solution:**

$$104,000 = 68,000(1+r)^{17} \rightarrow$$

1. Substitute in compound interest formula.

$$\frac{104,000}{68,000} = (1+r)^{17} \rightarrow$$

2. Divide both sides by 68,000

$$\sqrt[17]{\frac{104,000}{68,000}} = (1+r) \rightarrow$$

3. Take the 17th root of both sides of equation

4. Subtract 1 from both sides to solve for r .

$$\sqrt[17]{\frac{104,000}{68,000}} - 1 = r = 0.0253$$

Computing the Inflation Rate (continued)



- If the inflation rate remains the same for the next 10 years, what will the house be worth in the year 2014?

Computing the Inflation Rate (continued)

- If the inflation rate remains the same for the next 10 years, what will the house be worth in the year 2014?
- **Solution:** From 1987 to 2014 is a period of 27 years. If the inflation rate stays the same over that period, $r = 0.0253$. Substituting into the compound interest formula, we have

$$A = 68,000(1 + 0.0253)^{27} = 133,501$$

Growth Time of an Investment



- How long will it take for \$5,000 to grow to \$15,000 if the money is invested at 8.5% compounded quarterly?

Growth Time of an Investment Solution

- How long will it take for \$5,000 to grow to \$15,000 if the money is invested at 8.5% compounded quarterly?

- Solution:**

1. Substitute values in the compound interest formula. $\rightarrow 15000 = 5000\left(1 + \frac{0.085}{4}\right)^{4t}$

2. divide both sides by 5,000 $\rightarrow 3 = 1.02125^{4t}$

3. Take the natural logarithm of both sides. $\rightarrow \ln(3) = \ln 1.02125^{4t}$

4. Use the exponent property of logarithms $\rightarrow \ln(3) = 4t \cdot \ln 1.02125$

5. Solve for t . $\rightarrow t = \frac{\ln(3)}{4 \ln(1.02125)} = 13.0617$

Continuous Compound Interest



Previously we indicated that increasing the number of compounding periods while keeping the interest rate, principal, and time constant resulted in a somewhat higher compounded amount. What would happen to the amount if interest were compounded daily, or every minute, or every second?

Answer to Continuous Compound Interest Question



As the number m of compounding periods per year increases without bound, the compounded amount approaches a limiting value. This value is given by the following formula:

$$\lim_{m \rightarrow \infty} P\left(1 + \frac{r}{m}\right)^{mt} = Pe^{rt}$$

$$A = Pe^{rt}$$

Here A is the compounded amount.

Example of Continuously Compounded Interest



What amount will an account have after 10 years if \$1,500 is invested at an annual rate of 6.75% compounded continuously?

Example of Continuously Compounded Interest

What amount will an account have after 10 years if \$1,500 is invested at an annual rate of 6.75% compounded continuously?

Solution: Use the continuous compound interest formula

$A = Pe^{rt}$ with $P = 1500$, $r = 0.0675$, and $t = 10$:

$$A = 1500e^{(0.0675)(10)} = \$2,946.05$$

That is only 18 cents more than the amount you receive by daily compounding.

Annual Percentage Yield

- The simple interest rate that will produce the same amount as a given compound interest rate in 1 year is called the **annual percentage yield (APY)**. To find the **APY**, proceed as follows:

Amount at simple interest APY after one year

= Amount at compound interest after one year

$$P(1 + APY) = P\left(1 + \frac{r}{m}\right)^m \rightarrow$$

$$1 + APY = \left(1 + \frac{r}{m}\right)^m \rightarrow$$

$$APY = \left(1 + \frac{r}{m}\right)^m - 1$$

This is also called the **effective rate**.

Annual Percentage Yield Example



What is the annual percentage yield for money that is invested at 6% compounded monthly?

Annual Percentage Yield Example



What is the annual percentage yield for money that is invested at 6% compounded monthly?

General formula:

$$APY = \left(1 + \frac{r}{m}\right)^m - 1$$

Substitute values:

$$APY = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.06168$$

Effective rate is $0.06168 = 6.168$

Computing the Annual Nominal Rate Given the APY



- What is the annual nominal rate compounded monthly for a CD that has an annual percentage yield of 5.9%?

Computing the Annual Nominal Rate Given the APY

- What is the annual nominal rate compounded monthly for a CD that has an annual percentage yield of 5.9%?
- 1. Use the general formula for APY.
- 2. Substitute value of APY and 12 for m (number of compounding periods per year).
- 3. Add one to both sides
- 4. Take the twelfth root of both sides of equation.
- 5. Isolate r (subtract 1 and then multiply both sides of equation by 12).

$$APY = \left(1 + \frac{r}{m}\right)^m - 1$$

$$0.059 = \left(1 + \frac{r}{12}\right)^{12} - 1$$

$$1.059 = \left(1 + \frac{r}{12}\right)^{12}$$

$$\sqrt[12]{1.059} = \left(1 + \frac{r}{12}\right)$$

$$\sqrt[12]{1.059} - 1 = \frac{r}{12}$$

$$12\left(\sqrt[12]{1.059} - 1\right) = r$$

$$0.057 = r$$