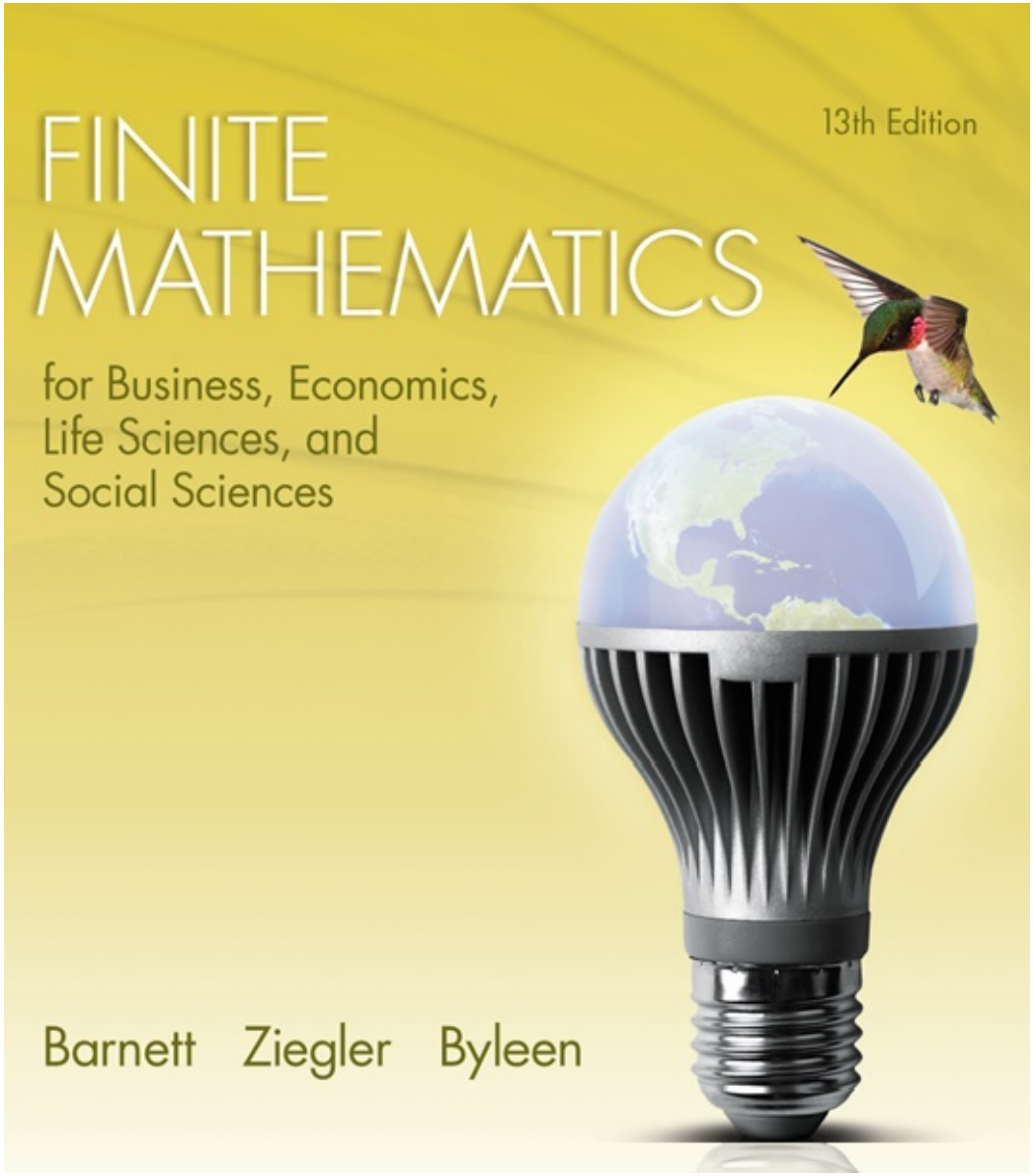

Chapter 3

Mathematics of Finance

Section 3

Future Value of an Annuity; Sinking Funds

Learning Objectives for Section 3.3

Future Value of an Annuity: Sinking Funds

- The student will be able to compute the future value of an annuity.
- The student will be able to solve problems involving sinking funds.
- The student will be able to approximate interest rates of annuities.

Definition of Annuity

- An **annuity** is any sequence of equal periodic payments.
- An **ordinary annuity** is one in which payments are made at the end of each time interval. If for example, $100 is deposited into an account every quarter (3 months) at an interest rate of 8% per year, the following sequence illustrates the growth of money in the account after one year:

$$100 + 100\left(1 + \frac{0.08}{4}\right) + 100(1.02)(1.02) + 100(1.02)(1.02)(1.02)$$

$$100 + \underbrace{100(1.02)}_{3^{rd}\text{ qtr}} + \underbrace{100(1.02)^2}_{2^{nd}\text{ quarter}} + \underbrace{100(1.02)^3}_{1^{st}\text{ quarter}}$$

This amount was just put in at the end of the 4th quarter, so it has earned no interest.

General Formula for Future Value of an Annuity

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

where

FV = future value (amount)

PMT = periodic payment

i = rate per period

n = number of payments (periods)

Note: Payments are made at the end of each period.

Example

- Suppose a $1000 payment is made at the end of each quarter and the money in the account is compounded quarterly at 6.5% interest for 15 years. How much is in the account after the 15 year period?

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$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

- **Solution:**

$$FV = 1000\left(\frac{\left(1+\frac{0.065}{4}\right)^{4(15)} - 1}{\frac{0.065}{4}}\right) = 100,336.68$$

Amount of Interest Earned

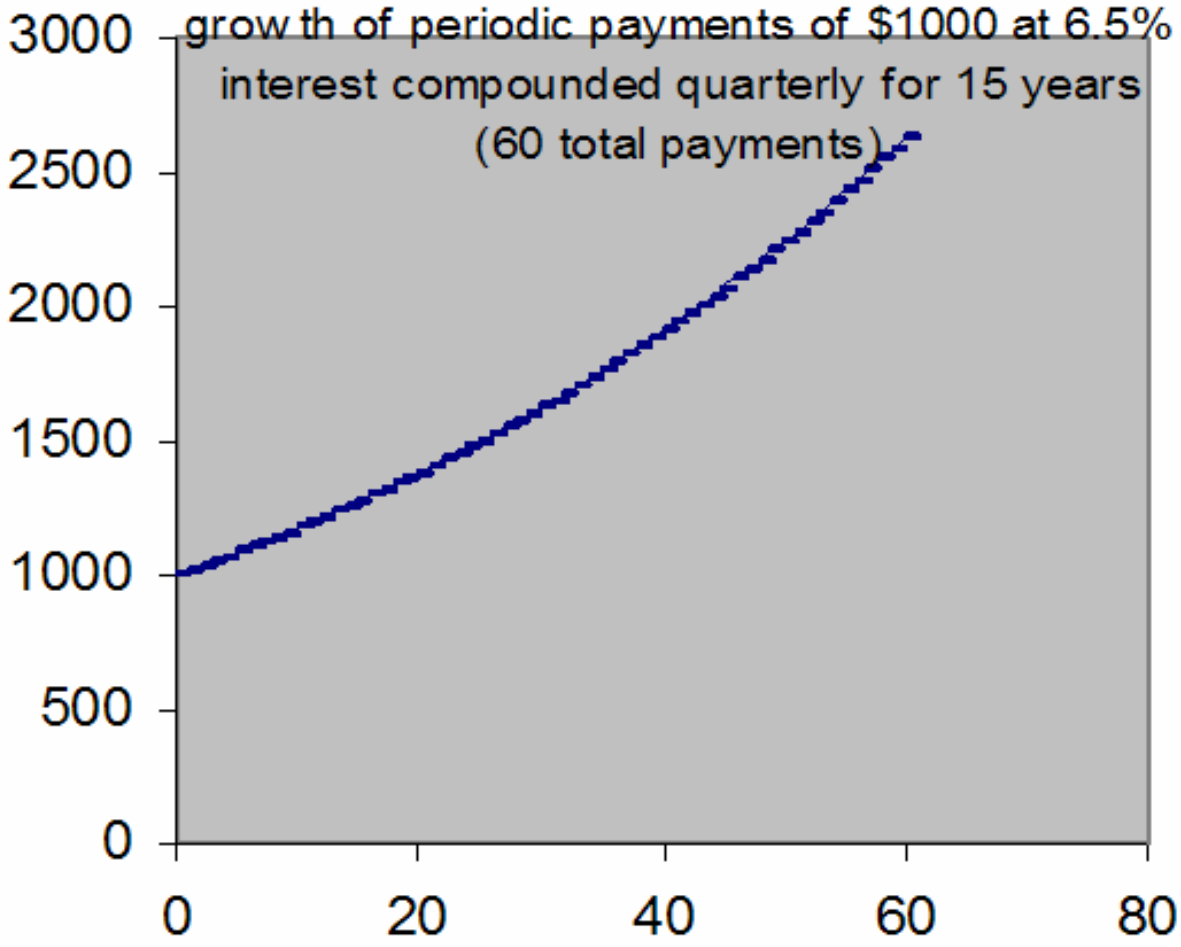
- How much interest was earned over the 15 year period?

Amount of Interest Earned Solution

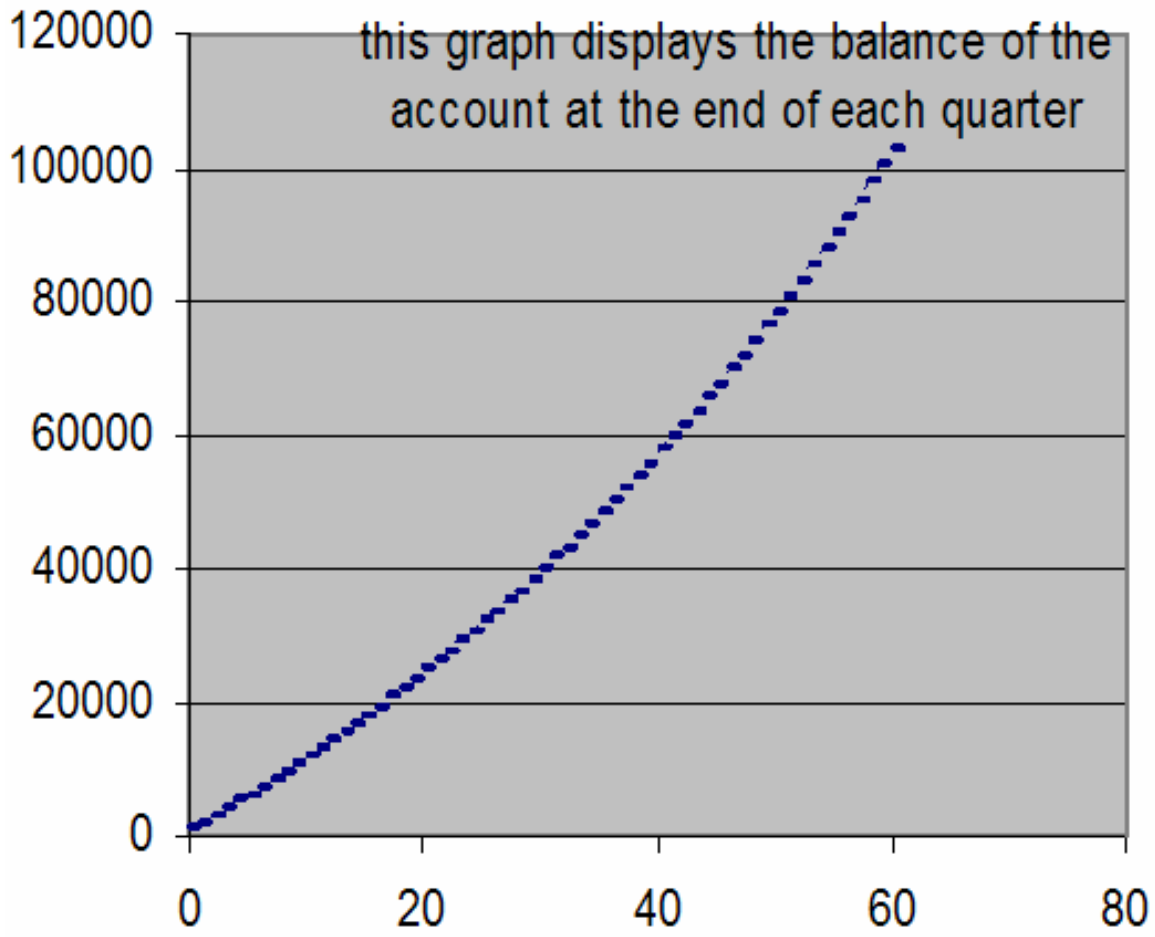
- How much interest was earned over the 15 year period?
- **Solution:**

  Each periodic payment was $1000. Over 15 years, 15(4)= 60 payments were made for a total of $60,000. Total amount in account after 15 years is $100,336.68. Therefore, amount of accrued interest is $100,336.68 − $60,000 = $40,336.68.

Graphical Display



Balance in the Account at the End of Each Period



Sinking Fund

- **Definition:** Any account that is established for accumulating funds to meet future obligations or debts is called a **sinking fund**.
- The **sinking fund payment** is defined to be the amount that must be deposited into an account periodically to have a given future amount.

Sinking Fund Payment Formula

To derive the sinking fund payment formula, we use algebraic techniques to rewrite the formula for the future value of an annuity and solve for the variable PMT:

$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

$$FV\left(\frac{i}{(1+i)^n - 1}\right) = PMT$$

Sinking Fund Sample Problem

How much must Harry save each month in order to buy a new car for $12,000 in three years if the interest rate is 6% compounded monthly?

Sinking Fund
Sample Problem Solution

How much must Harry save each month in order to buy a new car for $12,000 in three years if the interest rate is 6% compounded monthly?

Solution:

$$FV\left(\frac{i}{(1+i)^n-1}\right)=PMT$$

$$12000\left(\frac{\frac{0.06}{12}}{\left(1+\frac{0.06}{12}\right)^{36}-1}\right)=pmt=305.06$$

Approximating Interest Rates Example

Mr. Ray has deposited $150 per month into an ordinary annuity. After 14 years, the annuity is worth $85,000. What annual rate compounded monthly has this annuity earned during the 14 year period?

Approximating Interest Rates Solution

Mr. Ray has deposited $150 per month into an ordinary annuity. After 14 years, the annuity is worth $85,000. What annual rate compounded monthly has this annuity earned during the 14 year period?

Solution: Use the *FV* formula: Here *FV* = $85,000, *PMT* = $150 and *n*, the number of payments is 14(12) = 168. Substitute these values into the formula. Solution is approximated graphically.

Solution (continued)

$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

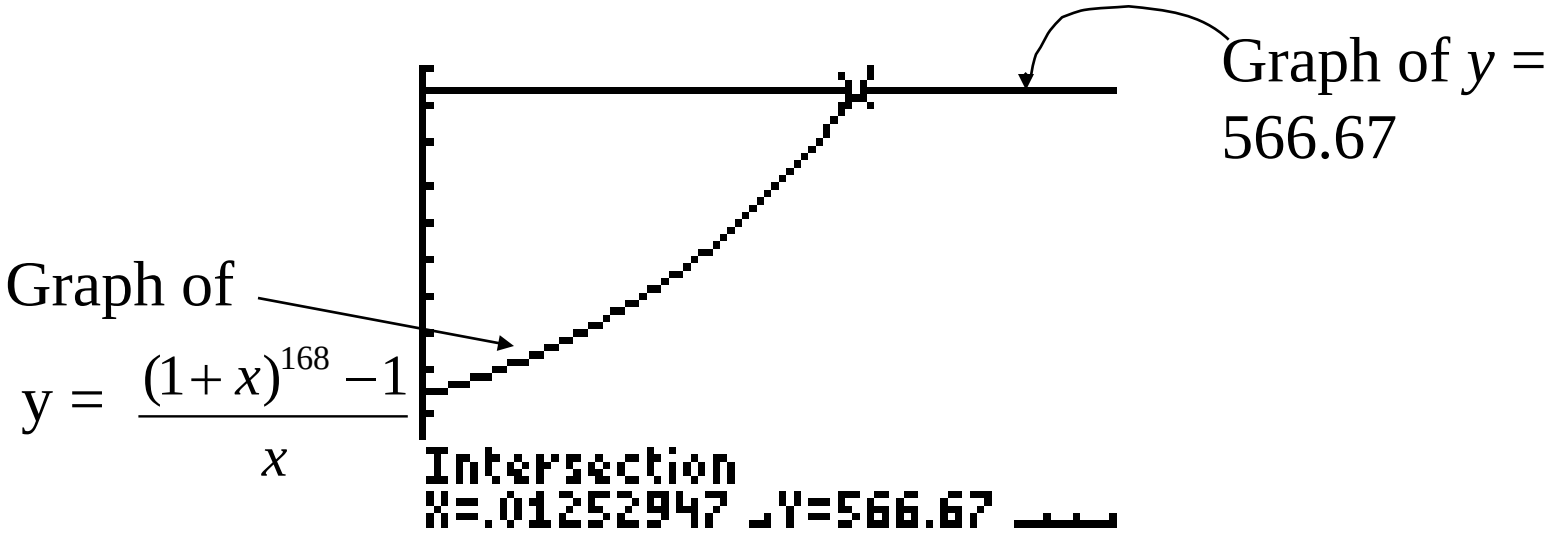
$$85,000 = 150\left(\frac{(1+i)^{14(12)} - 1}{i}\right)$$

$$\frac{85,000}{150} = \left(\frac{(1+i)^{168} - 1}{i}\right)$$

$$y = \left(\frac{(1+x)^{168} - 1}{x}\right) = \frac{85,000}{150} = 566.67$$

Graph each side of the last equation separately on a graphing calculator and find the point of intersection.

Solution (continued)



The monthly interest rate is about 0.01253 or 1.253%.
The annual interest rate is about 15%.