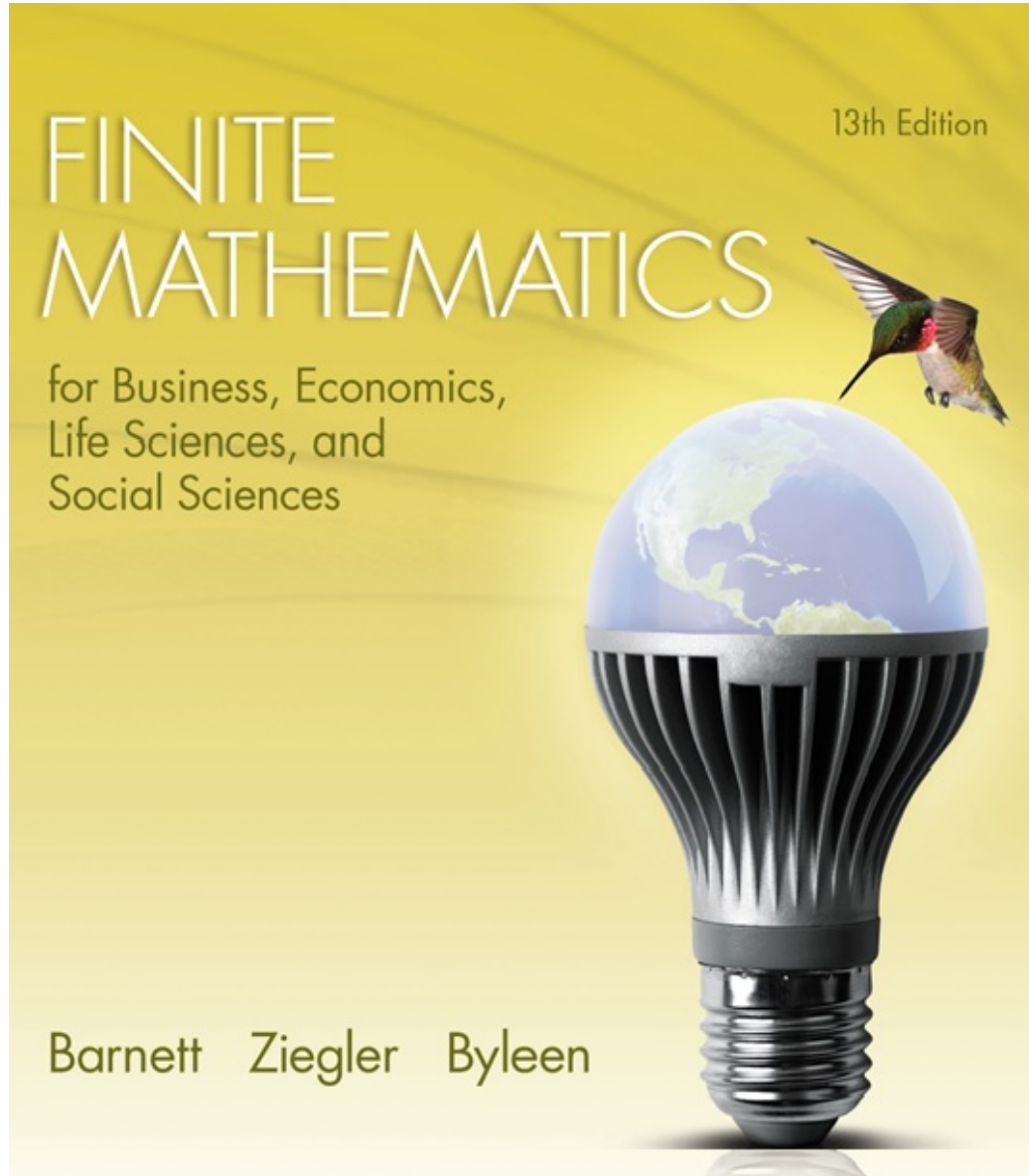




Chapter 3

Mathematics of Finance

Section 4

Present Value of an Annuity; Amortization

Learning Objectives for Section 3.4



Present Value of an Annuity; Amortization

- The student will be able to calculate the present value of an annuity.
- The student will be able to construct amortization schedules.
- The student will be able to calculate the payment for a loan.
- The student will have developed a strategy for solving mathematics of finance problems.

Present Value of an Annuity



- In this section, we will address the problem of determining the amount that should be deposited into an account now at a given interest rate in order to be able to withdraw equal amounts from the account in the future until no money remains in the account.
- Here is an example: How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw \$3,000 at the end of each quarter year for two years?

Derivation of Formula



- We begin by solving for P in the compound interest formula:

$$A = P(1 + i)^n \rightarrow$$

$$P = A(1 + i)^{-n}$$

Present Value of the First Four Payments



- Interest rate each period is $0.06/4=0.015$

$$P_1 = 3000 \left(1 + \frac{0.06}{4} \right)^{-1}$$

$$P_2 = 3000 (1.015)^{-2}$$

$$P_3 = 3000 (1.015)^{-3}$$

$$P_4 = 3000 (1.015)^{-4}$$

Derivation of Short Cut Formula



- We could proceed to calculate the next four payments and then simply find the total of the 8 payments. There are 8 payments since there will be 8 total withdrawals:
 $(2 \text{ years}) \times (\text{four withdrawals per year}) = 8 \text{ withdrawals.}$

This method is tedious and time consuming so we seek a short cut method.

Present Value of an Ordinary Annuity



$$PV = PMT \frac{1 - (1 + i)^{-n}}{i}$$

PV = present value of all payments

PMT = periodic payment

i = rate per period

n = number of periods

Note: Payments are made at the end of each period.

Back to Our Original Problem



- How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw \$3,000 at the end of each quarter year for two years?

Back to Our Original Problem

- How much money must you deposit now at 6% interest compounded quarterly in order to be able to withdraw \$3,000 at the end of each quarter year for two years?
- **Solution:** $R = 3000$, $i = 0.06/4 = 0.015$, $n = 8$

$$P = R \left(\frac{1 - (1 + i)^{-n}}{i} \right) \rightarrow$$

$$P = 3000 \left(\frac{1 - (1.015)^{-8}}{0.015} \right) = 22,457.78$$

Interest Earned



- The present value of all payments is \$22,457.78. The total amount of money withdrawn over two years is

$$3000(4)(2)=24,000.$$

Thus, the accrued interest is the difference between the two amounts:

$$\$24,000 - \$22,457.78 = \$1,542.22.$$

Amortization Problem



- **Problem:** A bank loans a customer \$50,000 at 4.5% interest per year to purchase a house. The customer agrees to make monthly payments for the next 15 years for a total of 180 payments. How much should the monthly payment be if the debt is to be retired in 15 years?

Amortization Problem Solution

- **Problem:** A bank loans a customer \$50,000 at 4.5% interest per year to purchase a house. The customer agrees to make monthly payments for the next 15 years for a total of 180 payments. How much should the monthly payment be if the debt is to be retired in 15 years?
- **Solution:** The bank has bought an annuity from the customer. This annuity pays the bank a $\$PMT$ per month at 4.5% interest compounded monthly for 180 months.

Solution (continued)

- We use the previous formula for present value of an annuity and solve for PMT :

$$PV = PMT \left(\frac{1 - (1 + i)^{-n}}{i} \right) \rightarrow$$

$$PMT = PV \left(\frac{i}{1 - (1 + i)^{-n}} \right)$$

Solution (continued)

Care must be taken to perform the correct order of operations.

$$PMT = PV \left(\frac{i}{1 - (1+i)^{-n}} \right) \rightarrow$$

1. enter 0.045 divided by 12

2. 1 + step 1 result

3. Raise answer to -180 power.

4. 1 – step 3 result

5. Take reciprocal (1/x) of step 4 result. Multiply by 0.045 and divide by 12.

5. Finally, multiply that result by 50,000 to obtain 382.50

$$PMT = 50,000 \left(\frac{\frac{0.045}{12}}{1 - \left(1 + \frac{0.045}{12} \right)^{-180}} \right) = 382.50$$

Solution (continued)



- If the customer makes a monthly payment of \$382.50 to the bank for 180 payments, then the total amount paid to the bank is the product of \$382.50 and 180 = \$68,850. Thus, the interest earned by the bank is the difference between \$68,850 and \$50,000 (original loan) = \$18,850.

Constructing an Amortization Schedule



If you borrow \$500 that you agree to repay in six equal monthly payments at 1% interest per month on the unpaid balance, how much of each monthly payment is used for interest and how much is used to reduce the unpaid balance?

Amortization Schedule Solution

If you borrow \$500 that you agree to repay in six equal monthly payments at 1% interest per month on the unpaid balance, how much of each monthly payment is used for interest and how much is used to reduce the unpaid balance?

Solution: First, we compute the required monthly payment using the formula

$$\begin{aligned}PMT &= PV \frac{i}{1 - (1 + i)^{-n}} \\&= 500 \frac{0.01}{1 - (1.01)^{-6}} \\&= \$86.27\end{aligned}$$

Solution (continued)

At the end of the first month, the interest due is
 $\$500(0.01) = \5.00 .

The amortization payment is divided into two parts, payment of the interest due and reduction of the unpaid balance.

Monthly Payment		Interest Due		Unpaid Balance Reduction
\$86.27	=	\$5.00	+	\$81.27

The unpaid balance for the next month is

Previous Unpaid Bal		Unpaid Bal Reduction		New Unpaid Bal
\$500.00	–	\$81.27	=	\$418.73

Solution (continued)

This process continues until all payments have been made and the unpaid balance is reduced to zero. The calculations for each month are listed in the following table, which was done on a spreadsheet.

Payment Number	Payment	Interest	Inpaid Bal Reduction	Unpaid Balance
0				\$500.00
1	86.27	\$5.00	\$81.27	\$418.73
2	86.27	\$4.19	\$82.08	\$336.65
3	86.27	\$3.37	\$82.90	\$253.74
4	86.27	\$2.54	\$83.73	\$170.01
5	86.27	\$1.70	\$84.57	\$85.44
6	86.27	\$0.85	\$85.42	\$0.03

In reality, the last payment would be increased by \$0.03, so that the balance is zero.

Strategy for Solving Mathematics of Finance Problems



- **Step 1.** Determine whether the problem involves a single payment or a sequence of equal periodic payments.
 - Simple and compound interest problems involve a **single** present value and a single future value.
 - Ordinary annuities may be concerned with a present value or a future value but always involve a **sequence** of equal periodic payments.

Strategy (continued)

- **Step 2.** If a single payment is involved, determine whether **simple** or **compound** interest is used. Simple interest is usually used for durations of a year or less and compound interest for longer periods.
- **Step 3.** If a sequence of periodic payments is involved, determine whether the payments are being made into an account that is increasing in value - a **future value** problem - or the payments are being made out of an account that is decreasing in value - a **present value** problem. Remember that amortization problems always involve the present value of an ordinary annuity.