4.2. System of linear Equations and Augmented Matrices

Goals:
1. Understand matrix terminology.
2. Solve augmented matrix associated with a linear system of 2 variables
3. Identify 3 possible matrix types for a linear system of 2 variables

Matrix is a rectangular array of numbers

E.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

2-by-2 ; 3-by-2 ; 3-by-3

dimension of matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow \text{elements or entries of the matrix}$$

Augmented matrix associated with a linear system.

E.g.

$$x + 3y = 5$$
$$2x - y = 3$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right)$$

Augmented Matrix
Associated with this system

E.g.

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 7 \end{array}\right) \longrightarrow \begin{array}{l} x = 3 \\ y = 7 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} x + y = 3 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases}$$

$$x + y = 3$$
$$0 = 0$$

Infinite many solutions

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 2 \end{array}\right) \longrightarrow \begin{array}{l} x + y = 4 \\ 0 = 2 \end{array}$$

No solutions

Operations that produce row-equivalence matrices

1. Interchange any 2 rows.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & 3 & 5 \end{array}\right)$$

② Multiply a row by any nonzero constant.

E.g.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xleftrightarrow{4R_2} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 8 & -4 & 12 \end{array}\right)$$

③ Add a constant multiple of a row to another row

E.g.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xleftrightarrow{R_1 + 4R_2} \left(\begin{array}{cc|c} 9 & -1 & 17 \\ 2 & -1 & 3 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xleftrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right)$$

$$\xleftrightarrow{-\frac{1}{7}R_2}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - 3R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \begin{array}{l} x = 2 \\ y = 1 \end{array}$$

E.g.

$$\left(\begin{array}{cc|c} 10 & -2 & 6 \\ -5 & 1 & -3 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 5 & -1 & 3 \\ -5 & 1 & -3 \end{array}\right)$$

$$\Big\downarrow R_2 + R_1$$

$$\begin{array}{l} 5x - y = 3 \\ 0 = 0 \end{array} \longleftarrow \left(\begin{array}{cc|c} 5 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

Infinitely many solutions

E.g.

$$\begin{array}{l} 5x - 2y = -7 \\ -\frac{5}{2}x + y = 1 \end{array} \qquad \left(\begin{array}{cc|c} 5 & -2 & -7 \\ -\frac{5}{2} & 1 & 1 \end{array}\right)$$

$$\Big\downarrow R_1 + 2R_2$$

$$\left(\begin{array}{cc|c} 0 & 0 & -5 \\ -\frac{5}{2} & 1 & 1 \end{array}\right) \longrightarrow \text{No Solution.}$$

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3 possible final matrix form for a linear system of 2 variables

I $\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right) \longrightarrow$ unique solution
$x = a \; ; \; y = b$

II $\left(\begin{array}{cc|c} 1 & m & a \\ 0 & 0 & 0 \end{array}\right) \longrightarrow$ infinitely many solutions

III

$$\left(\begin{array}{cc|c} 1 & m & a \\ 0 & 0 & b \end{array}\right) \longrightarrow \text{No solutions}$$

$$b \neq 0$$