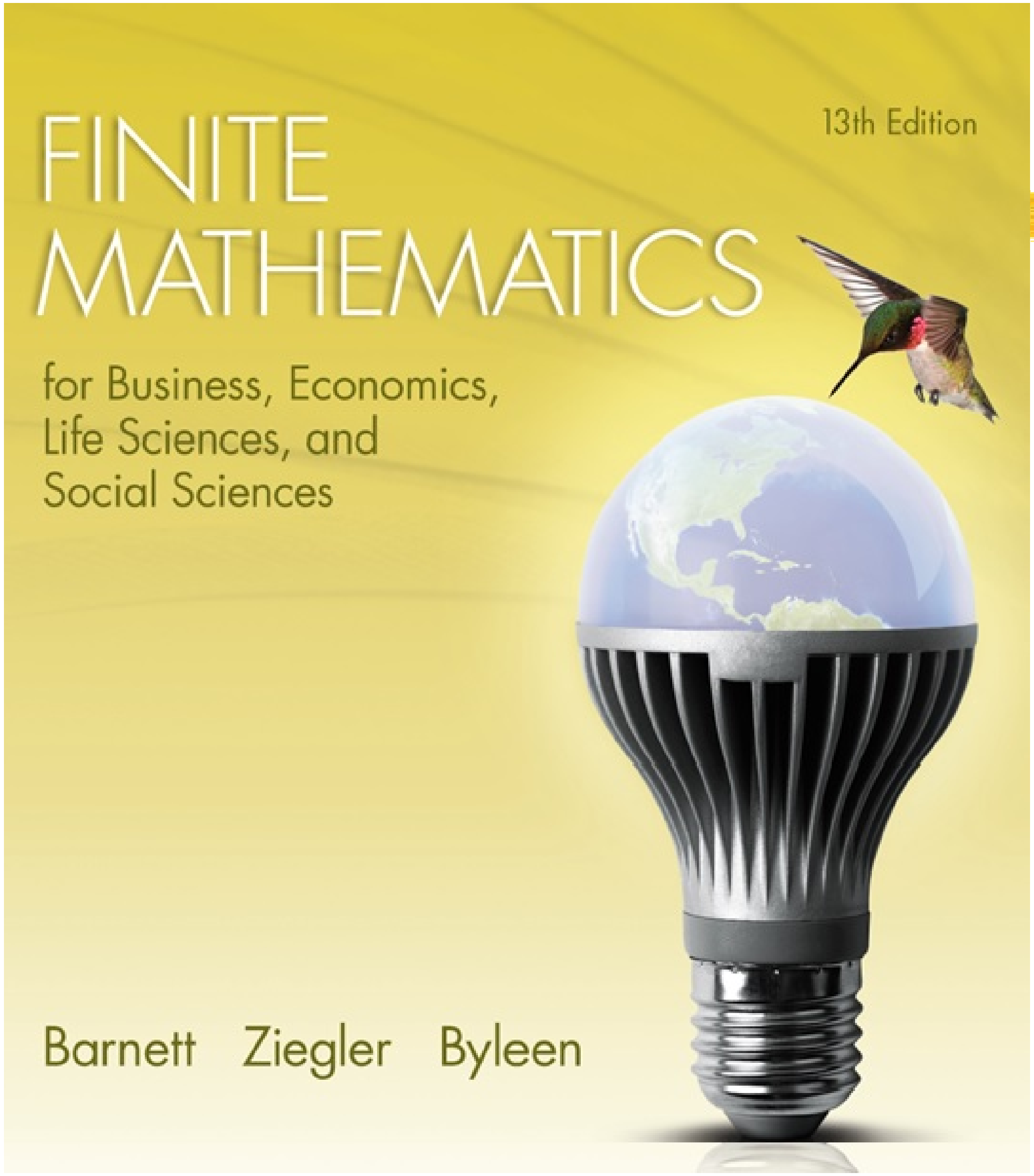


Chapter 4

Systems of Linear Equations; Matrices

Section 2

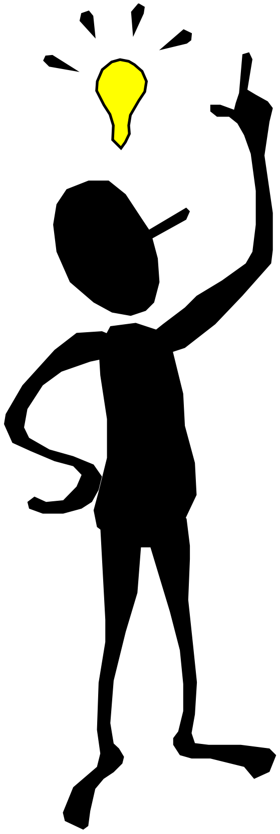
Systems of Linear Equations and Augmented Matrices

Learning Objectives for Section 4.2

Systems of Linear Equations and Augmented Matrices

- The student will be able to use terms associated with matrices.
- The student will be able to set up and solve the augmented matrix associated with a linear system in two variables.
- The student will be able to identify the three possible matrix solution types for a linear system in two variables.

Matrices



It is impractical to solve more complicated linear systems by hand. Computers and calculators now have built in routines to solve larger and more complex systems. Matrices, in conjunction with graphing utilities and or computers are used for solving more complex systems. In this section, we will develop certain matrix methods for solving two by two systems.

Matrices

A **matrix** is a rectangular array of numbers written within brackets. Here is an example of a matrix which has three rows and three columns: The subscripts give the "address" of each entry of the matrix. For example the entry a_{23} is found in the second row and third column

Since this matrix has 3 rows and 3 columns, the **dimensions** of the matrix are 3 x 3.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Each number in the matrix is called an **element**.

Matrix Solution of Linear Systems

When solving systems of linear equations, we can represent a linear system of equations by an **augmented matrix**, a matrix which stores the coefficients and constants of the linear system and then manipulate the augmented matrix to obtain the solution of the system.

Example:

$$x + 3y = 5$$

$$2x - y = 3$$

The augmented matrix associated with the above system is

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right]$$

Generalization

- Linear system:

$$a_{11}x_1 + a_{12}x_2 = k_1$$
$$a_{21}x_1 + a_{22}x_2 = k_2$$

- Associated augmented matrix:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

Operations that Produce Row-Equivalent Matrices

- 1. Two rows are interchanged: $R_i \leftrightarrow R_j$
- 2. A row is multiplied by a nonzero constant: $kR_i \to R_i$
- 3. A constant multiple of one row is added to another row: $kR_j + R_i \to R_i$

Note: The arrow $\to$ means "replaces."

Augmented Matrix Method Example

Solve

$$x + 3y = 5$$
$$2x - y = 3$$

1. Augmented system
2. Eliminate 2 in 2nd row by row operation
3. Divide row two by –7 to obtain a coefficient of 1.
4. Eliminate the 3 in first row, second position.
5. Read solution from matrix

▪ :

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right]$$

$$R_2 / -7 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$-3R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \to x = 2, y = 1; (2,1)$$

Augmented Matrix Method Example

Solve

$$x + 2y = 4$$
$$x + (1/2)y = 4$$

- Eliminate fraction in second equation by multiplying by 2
- Write system as augmented matrix.
- Multiply row 1 by –2 and add to row 2
- Divide row 2 by –3
- Multiply row 2 by –2 and add to row 1.
- Read solution : $x = 4$, $y = 0$
- (4,0)

$$x + 2y = 4$$

$$x + \frac{1}{2}y = 4 \rightarrow 2x + y = 8$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 8 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \end{array}\right] \rightarrow$$

Augmented Matrix Method Example

Solve

$$10x - 2y = 6$$

$$-5x + y = -3$$

1. Represent as augmented matrix. $\left[\begin{array}{cc|c} 10 & -2 & 6 \\ -5 & 1 & -3 \end{array}\right]$
2. Divide row 1 by 2 $\left[\begin{array}{cc|c} 5 & -1 & 3 \\ -5 & 1 & -3 \end{array}\right]$
3. Add row 1 to row 2 and replace row 2 by sum $\left[\begin{array}{cc|c} 5 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$
4. Since 0 = 0 is always true, we have a dependent system. The two equations are identical, and there are infinitely many solutions.

Augmented Matrix Method Example

- Solve $\begin{aligned} 5x - 2y &= -7 \\ y &= \frac{5}{2}x + 1 \end{aligned}$
- Rewrite second equation
- Add first row to second row
- The last row is the equivalent of $0x + 0y = -5$
- Since we have an impossible equation, there is no solution. The two lines are parallel and do not intersect.

$$\left.\begin{aligned} 5x - 2y &= -7 \\ -5x + 2y &= 2 \end{aligned}\right\} \rightarrow$$

$$\left[\begin{array}{cc|c} 5 & -2 & -7 \\ -5 & 2 & 2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|c} 5 & -2 & -7 \\ 0 & 0 & -5 \end{array}\right]$$

Possible Final Matrix Forms for a Linear System in Two Variables

Form 1: Unique Solution
(Consistent and Independent)

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

Form 2: Infinitely Many Solutions
(Consistent and Dependent)

$$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$$

Form 3: No Solution (Inconsistent)

$$\left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$$