FINITE MATHEMATICS >

for Business, Economics, Life Sciences, and Social Sciences

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13th Edition

Chapter 4

Systems of Linear Equations; Matrices

Section 3 Gauss-Jordan Elimination

ALWAYS LEARNING

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Learning Objectives for Section 4.3 Gauss-Jordan Elimination

- The student will be able to convert a matrix to reduced row echelon form.
- The student will be able to solve systems by Gauss-Jordan elimination.
- The student will be able to solve applications using Gauss-Jordan elimination.



Gauss-Jordan Elimination

- Any linear system must have exactly one solution, no solution, or an infinite number of solutions.
- Previously we considered the 2 × 2 case, in which the term consistent is used to describe a system with a unique solution, inconsistent is used to describe a system with no solution, and dependent is used for a system with an infinite number of solutions. In this section we will consider larger systems with more variables and more equations, but the same three terms are used to describe them.

Carl Friedrich Gauss 1777-1855

At the age of seven, Carl **Friedrich Gauss** started elementary school, and his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers, each pair summing to 101.



Matrix Representations of Consistent, Inconsistent and Dependent Systems

- The following matrix representations of three linear equations in three unknowns illustrate the three different cases:
 - Case I: consistent $\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 5 \end{pmatrix}$

From this matrix representation, you can determine that x = 3

$$y = 4$$
$$z = 5$$

Matrix Representations (continued)



From the second row of the matrix, we find that 0x + 0y + 0z = 6

or

0 = 6,

an impossible equation. From this, we conclude that there are no solutions to the linear system.

Matrix Representations (continued)

Case 3: dependent

$$\begin{pmatrix} 1 & 2 & 3 & | \ 4 \\ 0 & 0 & 0 & | \ 0 \\ 0 & 0 & 0 & | \ 0 \end{pmatrix}$$

 When there are fewer non-zero rows of a system than there are variables, there will be infinitely many solutions, and therefore the system is dependent.

Reduced Row Echelon Form

- A matrix is said to be in **reduced row echelon form** or, more simply, in **reduced form**, if
 - Each row consisting entirely of zeros is below any row having at least one non-zero element.
 - The leftmost nonzero element in each row is 1.
 - All other elements in the column containing the leftmost 1 of a given row are zeros.
 - The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Examples of Reduced Row Echelon Form

$$\begin{pmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 0 & 0 & | & 6 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & | -2 \\ 0 & 0 & 1 & 0 & | 7 \\ 0 & 0 & 0 & 1 & | 8 \end{pmatrix}$$

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Solving a System Using Gauss-Jordan Elimination

• **Example:** Solve

$$x + y - z = -2$$
$$2x - y + z = 5$$
$$-x + 2y + 2z = 1$$

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Solving a System Using Gauss-Jordan Elimination

• Example: Solve

$$x + y - z = -2$$

$$2x - y + z = 5$$

$$-x + 2y + 2z = 1$$

Solution:

We begin by writing the system as an augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & | -2 \\ 2 & -1 & 1 & | 5 \\ -1 & 2 & 2 & | 1 \end{bmatrix}$$

We already have a 1 in the diagonal position of first column. Now we want 0's below the 1. The first 0 can be obtained by multiplying row 1 by -2 and adding the results to row 2:

- Row 1 is unchanged
- (-2) times Row 1 is added to Row 2
- Row 3 is unchanged

$$\begin{bmatrix} 1 & 1 & -1 & | & -2 \\ 2 & -1 & 1 & | & 5 \\ -1 & 2 & 2 & | & 1 \end{bmatrix} \Rightarrow$$
$$\begin{bmatrix} 1 & 1 & -1 & | & -2 \\ 0 & -3 & 3 & | & 9 \\ -1 & 2 & 2 & | & 1 \end{bmatrix}$$

The second 0 can be obtained by adding row 1 to row 3:

- Row 1 is unchanged
- Row 2 is unchanged
- Row 1 is added to Row 3



Moving to the second column, we want a 1 in the diagonal position (where there was a -3). We get this by dividing every element in row 2 by -3:

- Row 1 is unchanged
- Row 2 is divided by -3
- Row 3 is unchanged

[1	1	_	1	- 2	
0	_	3	3	9	
0	-	3	1	-1	
•			I	-	
[1	1	-1	-	-2 -	
0	1	-1		-3	
0	3	1		-1	

To obtain a 0 below the 1, we multiply row 2 by -3 and add it to the third row:

- Row 1 is unchanged
- Row 2 is unchanged
- (-3) times row 2 is added to row 3



To obtain a 1 in the third position of the third row, we divide that row by 4. Rows 1 and 2 do not change.



PEARSON

We can now work upwards to get zeros in the third column, above the 1 in the third row.

- Add R3 to R2 and replace R2 with that sum
- Add R3 to R1 and replace R1 with the sum.
- Row 3 will not be changed.

All that remains to obtain reduced row echelon form is to eliminate the 1 in the first row, 2nd position.



To get a zero in the first row and second position, we multiply row 2 by -1and add the result to row 1 and replace row 1 by that result. Rows 2 and 3 remain unaffected.



Final Result

We can now "read" our solution from this last matrix. We have

$$x = 1,$$

 $y = -1$
 $z = 2.$

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

Written as an ordered triple, we have (1, -1, 2). This is a **consistent** system with a unique solution.

Example



Example: Solve the system 3x - 4y + 4z = 7 x - y - 2z = 22x - 3y + 6z = 5





Example

Example: Solve the system 3x - 4y + 4z = 7 x - y - 2z = 22x - 3y + 6z = 5

Solution:

Begin by representing the system as an augmented matrix:



 Since the first number in the second row is a 1, we interchange rows 1 and 2 and leave row 3 unchanged:

$$\begin{pmatrix} 3 & -4 & 4 & | & 7 \\ 1 & -1 & -2 & | & 2 \\ 2 & -3 & 6 & | & 5 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -1 & -2 & | & 2 \\ 3 & -4 & 4 & | & 7 \\ 2 & -3 & 6 & | & 5 \end{pmatrix}$$

In this step, we will get zeros in the entries beneath the 1 in the first column:

- Multiply row 1 by -3, add to row 2 and replace row 2: $-3R1 + R2 \rightarrow R2$.
- Multiply row 1 by -2, add to row 3 and replace row 3: $-2R1+R3 \rightarrow R3$.

$\left(\right)$	1	-1	-2	2	
	3	-4	4	7	
	2	-3	6	5	
	(1	-1	-2	2	
	0	-1	10	1	
	0	-1	10	1	

Final Result

To get a zero in the third row, second entry we multiply row 2 by -1 and add the result to R3 and replace R3 by that sum: Notice this operations "wipes out" row 3 so row 3 consists entirely of zeros.

Any time you have fewer non-zero rows than variables you will have a **dependent** system.

Representation of a Solution of a Dependent System

$$\begin{pmatrix} 1 & -1 & -2 & | & 2 \\ 0 & -1 & 10 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

We can interpret the second row of this matrix as

$$-y + 10z = 1$$
, or
 $10z - 1 = y$

So, if we let z = t(arbitrary real number,) then in terms of t, y = 10t - 1. Next we can express the variable x in terms of t as follows: From the first row of the matrix, we have x - y - 2z = 2. If z = t and y = 10t - 1, we have x - (10t - 1) - 2t = 2 or x = 12t + 1. Our general solution can now be

(12t + 1, 10t - 1, t),

where *t* is an arbitrary real number.

expressed in terms of *t*:

Procedure for Gauss-Jordan Elimination

Step 1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.

Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Step 3. Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step and all rows above this row.

Step 4. Repeat step 2 with the entire matrix, including the rows deleted mentally. Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop because we will have a contradiction.

Applications

Systems of linear equations provide an excellent opportunity to discuss mathematical modeling. The process of using mathematics to solve real-world problems can be broken down into three steps:

Step 1. **Construct** a mathematical model whose solution will provide information about the real-world problem.



Applications (continued)

Step 2. Solve the mathematical model.

Step 3. **Interpret** the solution to the mathematical model in terms of the original real-world problem.

Example: Purchasing. A company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic feet. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many of each type of truck should the company purchase?

Solution

Step 1. The question in this example indicates that the relevant variables are the number of each type of truck.

x = number of 10-foot trucks

- y = number of 14-foot trucks
- z = number of 24-foot trucks

We form the mathematical model:

x + y + z = 25 (Total number of trucks)

350x + 700y + 1,400z = 28,000 (Total capacity)

Solution (continued)

Step 2. Now we form the augmented coefficient matrix of the system and solve by using Gauss-Jordan elimination:

 $\begin{bmatrix} 1 & 1 & 1 & 25 \\ 350 & 700 & 1,400 & 28,000 \end{bmatrix}$

 $\begin{vmatrix} 1 & 1 & 1 & 25 \\ 1 & 2 & 4 & 80 \end{vmatrix}$

 $\begin{bmatrix} 1 & 1 & 1 & | & 25 \\ 0 & 1 & 3 & | & 55 \end{bmatrix}$

 $\begin{bmatrix} 1 & 0 & -2 & | & -30 \\ 0 & 1 & 3 & | & 55 \end{bmatrix}$

$$(1/350)$$
R₂ \longrightarrow R₂

$$-\mathbf{R}_1 + \mathbf{R}_2 \longrightarrow \mathbf{R}_2$$

$$-\mathbf{R}_2 + \mathbf{R}_1 \longrightarrow \mathbf{R}_1$$

Matrix is in reduced form. x - 2z = -30 or x = 2z - 30, y + 3z = 55 or y = -3z + 55.

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Solution (continued)

Let z = t. Then for *t* any real number

$$x = 2t - 30$$
$$y = -3t + 55$$
$$z = t$$

is a solution to our mathematical model.

Step 3. We must interpret this solution in terms of the original problem. Since the variables x, y, and z represent numbers of trucks, they must be nonnegative. And since we can't purchase a fractional number of trucks, each must be a nonnegative whole number.

Solution (continued)

Since t = z, it follows that t must also be a nonnegative whole number. The first and second equations in the model place additional restrictions on the values t can assume:

 $x = 2t - 30 \ge 0$ implies that $t \ge 15$ $y = -3t + 55 \ge 0$ implies that $t \le 55/3$

Thus the only possible values of *t* that will produce meaningful solutions to the original problem are 15, 16, 17, and 18. A table is a convenient way to display these solutions.

	10-ft truck	14-ft. truck	24-ft truck
t	X	У	Z
15	0	10	15
16	2	7	16
17	4	4	17
18	6	1	18