# 4.5. The Inverse of a Square Matrix

**Goals:** (1) Find the inverse of a Square Matrix

(2) Application in Cryptography.

## Identity Matrix

The number 1 $\longrightarrow$ Multiplicative identity.

What matrix will play the role of the # 1 ?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{2-by-2 identity matrix}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

3-by-3 identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4-by-4 identity matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of a number: $5 \cdot \frac{1}{5} = 1$ (any nonzero # has an inverse)

What is the inverse of a matrix?

E.g. $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

To find the inverse of $A$, we need to find a matrix $B$ such that $AB = BA = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Formula to find the inverse of a 2-by-2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse of $A$ is given by the formula:

$$\boxed{B = A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Note: if $ad-bc=0$, $A$ does not have an inverse

E.g. $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. $B = A^{-1} = ?$

$$A^{-1} = \frac{1}{2\cdot 2 - 3\cdot 1}\cdot\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = 1\cdot\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

E.g.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2\cdot 8 - 3\cdot 4} \begin{pmatrix} 8 & -4 \\ -3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 8 & -4 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{3}{4} & \frac{1}{2} \end{pmatrix}$$

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Inverse Matrix of a 3-by-3

Process: Find the inverse of a 3-by-3 matrix

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$$

(1) Augmented the given matrix by the 3-by-3 identity matrix.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

(2) Apply row operations to this 3-by-6 matrix with the goal of turning the 3-by-3 submatrix on the left into the identity matrix

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & \boxed{2} & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -2 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+2R_1}\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -2 & 1 & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right)\xrightarrow{\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right)\xrightarrow{R_1+R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right)\xrightarrow{R_3+4R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} & -\frac{2}{3} & \frac{4}{3} & 1 \end{array}\right)\xrightarrow{\frac{3}{5}R_3}$$