6.2. Linear Programming. The Simplex Method.

Goal: Apply the Simplex method to solve maximization problem with constraints of the form $\leq$.

Recall: Last time we used the geometric approach to linear programming to solve the maximization

Maximize $P = 5x + 10y$ (profit)

Subject to constraints:

$$8x + 8y \leq 160 \quad ; \quad x, y \geq 0$$

$$4x + 12y \leq 180$$

<u>Recall</u>: (1) Graph the feasible region determined by the constraints

(2) Find the corner points of the feasible region.

(3) Plug the corner points into the objective function to find the optimal solution.

Solution was $x = 7.5$; $y = 12.5$, $P = 162.5$
max value

If the system is larger, i.e. if there are more variables, the geometric method may not work. $\longrightarrow$ the simplex method.

Step 1: Introduce Slack Variables

Original system:

$$8x + 8y \leq 160$$
$$4x + 12y \leq 180$$
$$P = 5x + 10y$$

Rename x into x_1
y into x_2

2 inequalities $\longrightarrow$ 2 slack variables s_1, s_2

$$8x_1 + 8x_2 + s_1 = 160$$
$$4x_1 + 12x_2 + s_2 = 180$$
$$-5x_1 - 10x_2 + P = 0$$
$$x_1, x_2, s_1, s_2 \geq 0$$

s_1, s_2, P : basic variables $\quad$ x_1, x_2 : non-basic variables

Step 2: Form the simplex tableau.

(3-by-5 matrix; augmented by the RHS)

# of basic variables

total # of variables

Pivot column

Pivot Row

$$
\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
\hline
s_1 & 8 & 8 & 1 & 0 & 0 & 160 \\
s_2 & 4 & 12 & 0 & 1 & 0 & 180 \\
P & -5 & -10 & 0 & 0 & 1 & 0
\end{array}
$$

160/8 = 20

180/12 = 15

Step 3: Find the pivot column and pivot row of the tableau → entering and exiting variable

Find pivot column.

Are there any negative # in the bottom row?
(If there are none, we are done! the rightmost column is the solution)

Yes $\longrightarrow$ find the most negative #.

$-10 \longrightarrow$ column 2 is the pivot column.

To determine the pivot row, divide the coefficients above -10 into the #'s in the right most column and find the smallest quotient

$\longrightarrow$ row 2 is the pivot row enter

pivot column

exit

$$
\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
\hline
s_1 & 8 & 8 & 1 & 0 & 0 & 160 \\
s_2 & 4 & 12 & 0 & 1 & 0 & 180 \\
P & -5 & -10 & 0 & 0 & 1 & 0
\end{array}
$$

pivot row

- The variable corr. to the pivot column is called the entering variable
- The one corr to the pivot row is the exiting variable

Step 4:

Step 4: Use basic row operations to obtain 1 in the pivot position and zeros in the other positions of the pivot column

$$\left(\begin{array}{ccccc|c} 8 & 8 & 1 & 0 & 0 & 160 \\ 4 & \boxed{12} & 0 & 1 & 0 & 180 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\substack{R_1 - 8R_2 \to R_1 \\ R_3 + 10R_2 \to R_3}]{\frac{1}{12}R_2 \to R_2}$$

exit

becomes enter

$$\begin{array}{c} \\ s_1 \\ x_2 \\ P \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \left(\begin{array}{ccccc|c} \frac{16}{3} & 0 & 1 & -\frac{2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ -\frac{5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

Step 5: If there are still negative # on the bottom row, repeat the process until there are no more negative # on the bottom row

Pivot row

$x_1 \to$ enter

Pivot position

$$
\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
s_1 \text{ (exit)} & \boxed{\frac{16}{3}} & 0 & 1 & -\frac{2}{3} & 0 & 40 \\
x_2 & \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\
P & -\frac{5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150
\end{array}
$$

$40 / \frac{16}{3} = 7.5$

$15 / \frac{1}{3} = 45$

Pivot column

$$\frac{3}{16} R_1 \to R_1$$

$$R_2 - \frac{1}{3} R_1 \to R_2$$

$$R_3 + \frac{5}{3} R_1 \to R_3$$

$$\begin{array}{c|cccccc}
 & x_1 & x_2 & s_1 & s_2 & P & \\
\hline
x_1 & 1 & 0 & 3/16 & -1/8 & 0 & 7.5 \\
x_2 & 0 & 1 & -1/16 & 1/8 & 0 & 12.5 \\
P & 0 & 0 & 5/16 & 5/8 & 1 & 162.5
\end{array}$$

Step 6: Once we got no more negative in the bottom row, the right most column gives us the optimal solution.

$x_1 = 7.5, \quad x_2 = 12.5, \quad P = 162.5$