6.3. The Dual Problem

Goal: Solve the minimization problem with constraints of the form $\geq$.

Recall: solve the maximization problem with constraints of the form $\leq$

Maximize $\boxed{P = 5x + 10y}$; $x, y \geq 0$

subject $8x + 8y \leq 160$

$4x + 12y \leq 180$

Now: Minimize cost $C = 16x_1 + 9x_2 + 21x_3$

subject $x_1 + x_2 + 3x_3 \geq 12$

$2x_1 + x_2 + x_3 \geq 16$

$x_1, x_2, x_3 \geq 0$

constraints

Key idea: translate this into the maximization problem subject $\leq$ constraints and use tableau method from last time

How do we do that?

Step 1: Write down the initial matrix for the problem.

$$A = \begin{pmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \end{pmatrix} \qquad 3\text{-by-}4$$

Step 2 Transpose the initial matrix

$$A^T = \begin{pmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{pmatrix} \qquad 4\text{-by-}3$$

Step 3: Use the transpose to rewrite the problem as a maximization problem (with variables $y_1, y_2, \ldots$)
(This is called forming the dual problem)

$$\left(\begin{array}{ccc} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{array}\right.$$

$$y_1 + 2y_2 \leq 16$$
$$y_1 + y_2 \leq 9$$
$$3y_1 + y_2 \leq 21$$
$$P = 12y_1 + 16y_2$$

$y_1, y_2 \geq 0$

to be maximized

Step 4: Apply the method from last time

→ Introduce slack variables (x_1, x_2, x_3, ...)

Write the tableau

Find pivot element

Do row operations.

Note: The solution (if any) of the original problem will be read off from the bottom row (instead of the right-most column)

We went through the process and obtained

$$
\begin{array}{c} \\ y_2 \\ y_1 \\ x_3 \\ P \end{array}
\begin{array}{c}
\begin{array}{cccccc|c} y_1 & y_2 & x_1 & x_2 & x_3 & P & \end{array} \\
\left(\begin{array}{cccccc|c}
0 & 1 & 1 & -1 & 0 & 0 & 7 \\
1 & 0 & -1 & 2 & 0 & 0 & 2 \\
0 & 0 & 2 & -5 & 1 & 0 & 8 \\
0 & 0 & 4 & 8 & 0 & 1 & 136
\end{array}\right)
\end{array}
\longrightarrow \text{Result}
$$

$\min C = 136$ when

$x_1 = 4$; $x_2 = 8$; $x_3 = 0$.