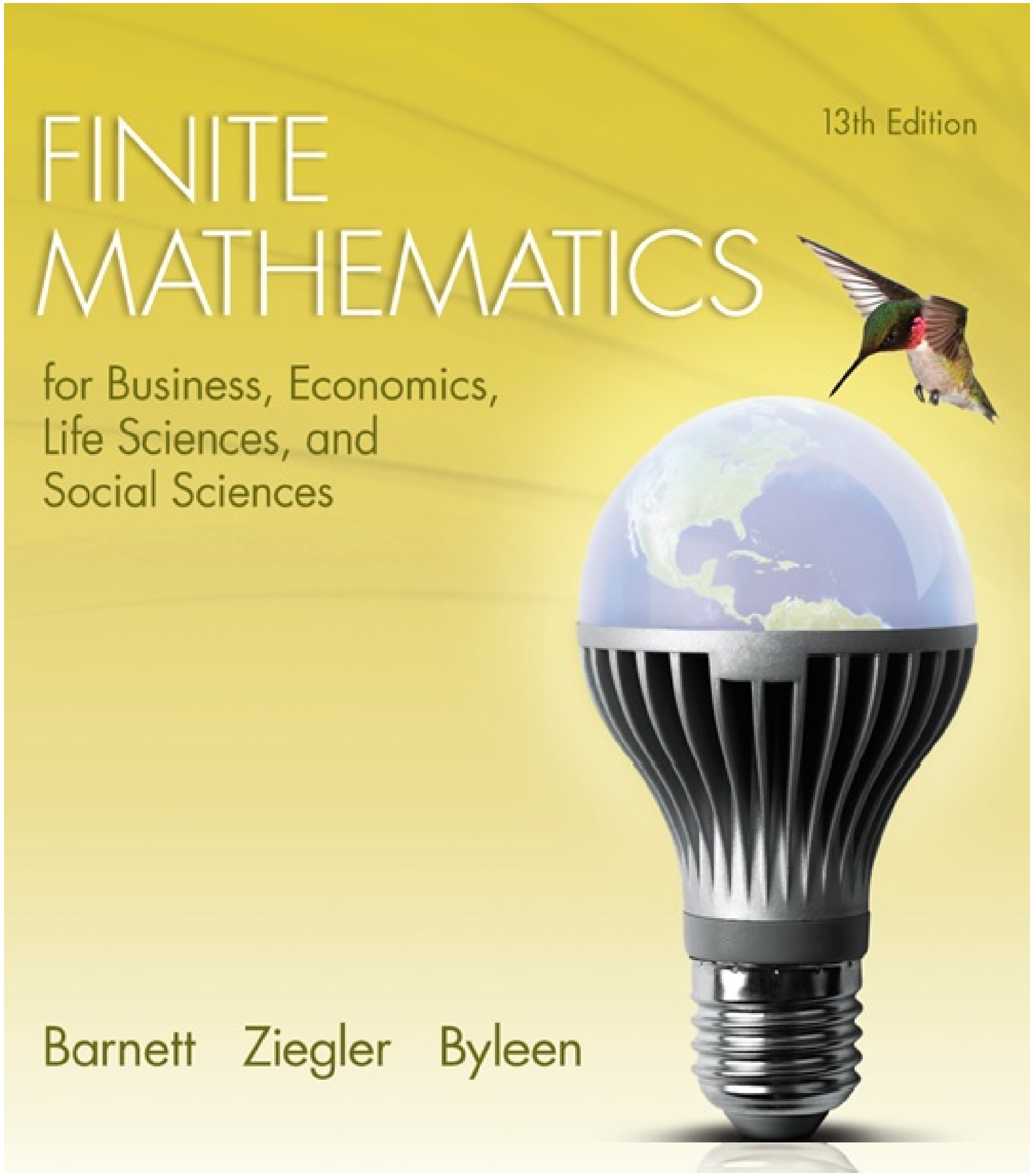


Chapter 7

Logic, Sets, and Counting

Section 4

Permutations and Combinations

Learning Objectives for Section 7.4 Permutations and Combinations

- The student will be able to set up and compute factorials.
- The student will be able to apply and calculate permutations.
- The student will be able to apply and calculate combinations.
- The student will be able to solve applications involving permutations and combinations.

7.4 Permutations and Combinations

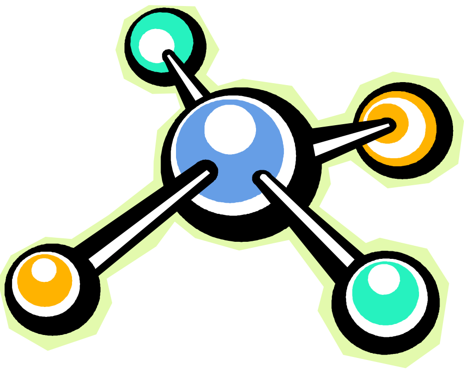
For more complicated problems, we will need to develop two important concepts: **permutations** and **combinations**. Both of these concepts involve what is called the **factorial** of a number.

Definition of *n* Factorial (*n* !)

- $n! = n(n-1)(n-2)(n-3)\ldots1$
  For example, $5! = 5(4)(3)(2)(1) = 120$
- $n! = n(n-1)!$
- $0! = 1$ by definition.
- Most calculators have an $n!$ key or the equivalent.
- $n!$ grows very rapidly, which may result in overload on a calculator.

Example

The simplest protein molecule in biology is called *vasopressin* and is composed of 8 amino acids that are chemically bound together in a particular order. The order in which these amino acids occur is of vital importance to the proper functioning of vasopressin. If these 8 amino acids were placed in a hat and drawn out randomly one by one, how many different arrangements of these 8 amino acids are possible?



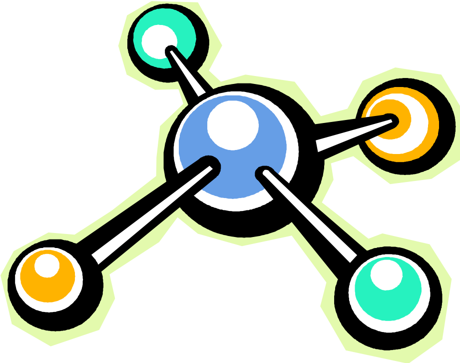
Solution

Solution: Let A, B, C, D, E, F, G, H symbolize the 8 amino acids. They must fill 8 slots:

___ ___ ___ ___ ___ ___ ___ ___

There are 8 choices for the first position, leaving 7 choices for the second slot, 6 choices for the third slot and so on. The number of different orderings is

$$8(7)(6)(5)(4)(3)(2)(1) = 8! = 40{,}320.$$



Two Problems Illustrating Combinations and Permutations

- **Problem 1:** Consider the set $\{p, e, n\}$. How many two-letter "words" (including nonsense words) can be formed from the members of this set, if two different letters have to be used?

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- **Solution:** We will list all possibilities: *pe*, *pn*, *en*, *ep*, *np*, *ne*, a total of 6.
- **Problem 2:** Now consider the set consisting of three males: {Paul, Ed, Nick}. For simplicity, denote the set by {*p*, *e*, *n*}. How many two-man crews can be selected from this set?

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- **Solution:** *pe* (Paul, Ed), *pn* (Paul, Nick) and *en* (Ed, Nick), and that is all!

Difference Between Permutations and Combinations

- Both problems involved counting the numbers of arrangements of the same set {*p*, *e*, *n*}, taken 2 elements at a time, without allowing repetition. However, in the first problem, the **order** of the arrangements **mattered** since pe and ep are two different “words”. In the second problem, the **order did not matter** since *pe* and *ep* represented the same two-man crew. We counted this only once.
- The first example was concerned with counting the number of **permutations** of 3 objects taken 2 at a time.
- The second example was concerned with the number of **combinations** of 3 objects taken 2 at a time.

Permutations

- The notation $P(n, r)$ represents the number of permutations (arrangements) of n objects taken r at a time, where r is less than or equal to n. **In a permutation, the order is important.**
- $P(n, r)$ may also be written $P_{n,r}$
- In our example with the number of two letter words from
  $\{p, e, n\}$, the answer is $P(3, 2)$, which represents the number of permutations of 3 objects taken 2 at a time. $P(3, 2) = 6 = (3)(2)$

Permutations

In general,

$$P(n, r) = n(n - 1)(n - 2)(n - 3)\ldots(n - r + 1)$$

or

$$P_{n,r} = \frac{n!}{(n-r)\ !} \qquad 0 \le r \le n$$

More Examples

- **Find *P*(5, 3)**

Here $n = 5$ and $r = 3$, so we have

$$P(5, 3) = (5)(5 - 1)(5 - 2) = 5(4)3 = 60.$$

This means there are 60 permutations of 5 items taken 3 at a time.

- **Application:** A park bench can seat 3 people. How many seating arrangements are possible if 3 people out of a group of 5 sit down?

More Examples (continued)

- **Solution:** Think of the bench as three slots

___ ___ ___ .

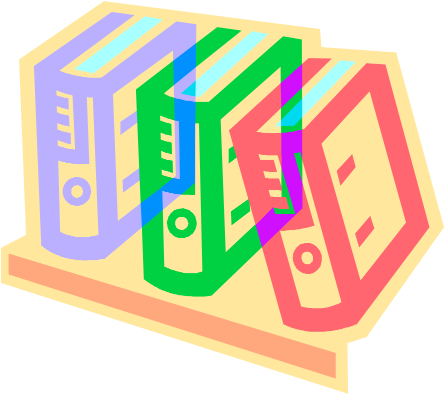
There are 5 people that can sit in the first slot, leaving 4 remaining people to sit in the second position and finally 3 people eligible for the third slot.

Thus, there are 5(4)(3) = 60 ways the people can sit.

The answer could have been found using the permutations formula $P(5, 3) = 60$, since we are finding the number of ways of arranging 5 objects taken 3 at a time.

$$P(n, n) = n(n-1)(n-2)\ldots 1 = n!$$

- Find $P(5, 5)$, the number of arrangements of 5 objects taken 5 at a time.
- **Answer:** $P(5, 5) = 5(4)(3)(2)(1) = 120$.
- **Application:** A bookshelf has space for exactly 5 books. How many different ways can 5 books be arranged on this bookshelf?



$P(n, n) = n(n-1)(n-2)\ldots 1 = n!$

- Find $P(5, 5)$, the number of arrangements of 5 objects taken 5 at a time.
- **Answer:** $P(5, 5) = 5(4)(3)(2)(1) = 120$.
- **Application:** A bookshelf has space for exactly 5 books. How many different ways can 5 books be arranged on this bookshelf?
- **Answer:** Think of 5 slots, again.
  There are five choices for the first slot, 4 for the second and so on until there is only 1 choice for the final slot. The answer is 5(4)(3)(2)(1), which is the same as P(5, 5) = 120.

Combinations

- In the second problem, the number of two man crews that can be selected from $\{p, e, n\}$ was found to be 6. This corresponds to the number of *combinations* of 3 objects taken 2 at a time or $C(3, 2)$. We will use a variation of the formula for permutations to derive a formula for combinations.
- Note: $C(n, r)$ may also be written $C_{n,r}$ or $\binom{n}{r}$.

Combinations

- Consider the six permutations of $\{p, e, n\}$ which are grouped in three pairs of 2. Each pair corresponds to one combination of 2:

$$(pe, ep), \quad (pn, np), \quad (en, ne)$$

- If we want to find the number of combinations of 3 objects taken 2 at a time, we simply divide the number of permutations of 3 objects taken 2 at a time by 2 (or 2!)
- We have the following result:

$$C(3, 2) = \frac{P(3,2)}{2!}$$

Generalization

General result: This formula gives the number of subsets of size *r* that can be taken from a set of *n* objects. The order of the items in each subset **does not matter.** The number of **combinations** of *n* distinct objects taken *r* at a time without repetition is given by

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{n!}{r!(n-r)!}$$

Examples
Solution

1. Find $C(8, 5)$

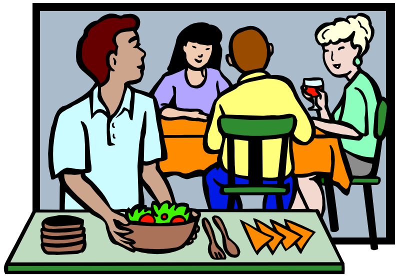
Solution: C(8, 5) = $\dfrac{8!}{5!(8-5)!} = \dfrac{8!}{5!3!} = 56$

2. Find C(8, 8)

Solution: C(8, 8) = $\dfrac{8!}{8!(8-8)!} = \dfrac{8!}{8!0!} = 1$

Combinations or Permutations? (continued)

- In how many ways can you choose 5 out of 10 friends to invite to a dinner party?



- **Solution:** Does the order of selection matter? If you choose friends in the order A,B,C,D,E or A,C,B,D,E, the same set of 5 was chosen, so we conclude that the **order of selection does not matter**. We will use the formula for **combinations** since we are concerned with how many subsets of size 5 we can select from a set of 10.

$$C(10,5) = \frac{10!}{5!(10-5)!} = \frac{10!}{5!5!} = 252$$

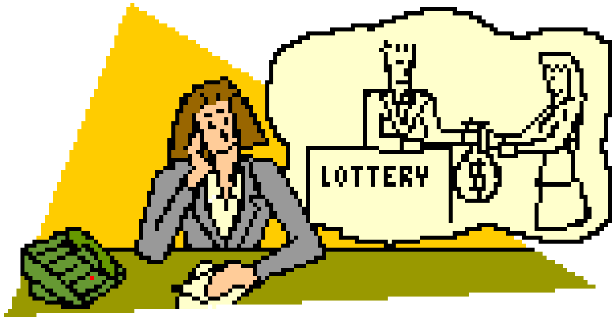
Permutations or Combinations? (continued)

- How many ways can you arrange 10 books on a bookshelf that has space for only 5 books?
- **Solution:** Does order matter? The answer is **yes** since the arrangement ABCDE is a different arrangement of books than BACDE. We will use the formula for **permutations**. We need to determine the number of arrangements of 10 objects taken 5 at a time so we have

$$P(10, 5) = 30{,}240.$$

Lottery Problem

- A certain state lottery consists of selecting a set of 6 numbers randomly from a set of 49 numbers. To win the lottery, you must select the correct set of six numbers. How many possible lottery tickets are there?



Lottery Problem

Solution: The **order of the numbers is not important** here as long as you have the correct set of six numbers. To determine the total number of lottery tickets, we will use the formula for **combinations** and find C(49, 6), the number of combinations of 49 items taken 6 at a time. Using our calculator, we find that C(49, 6) = 13,983,816.