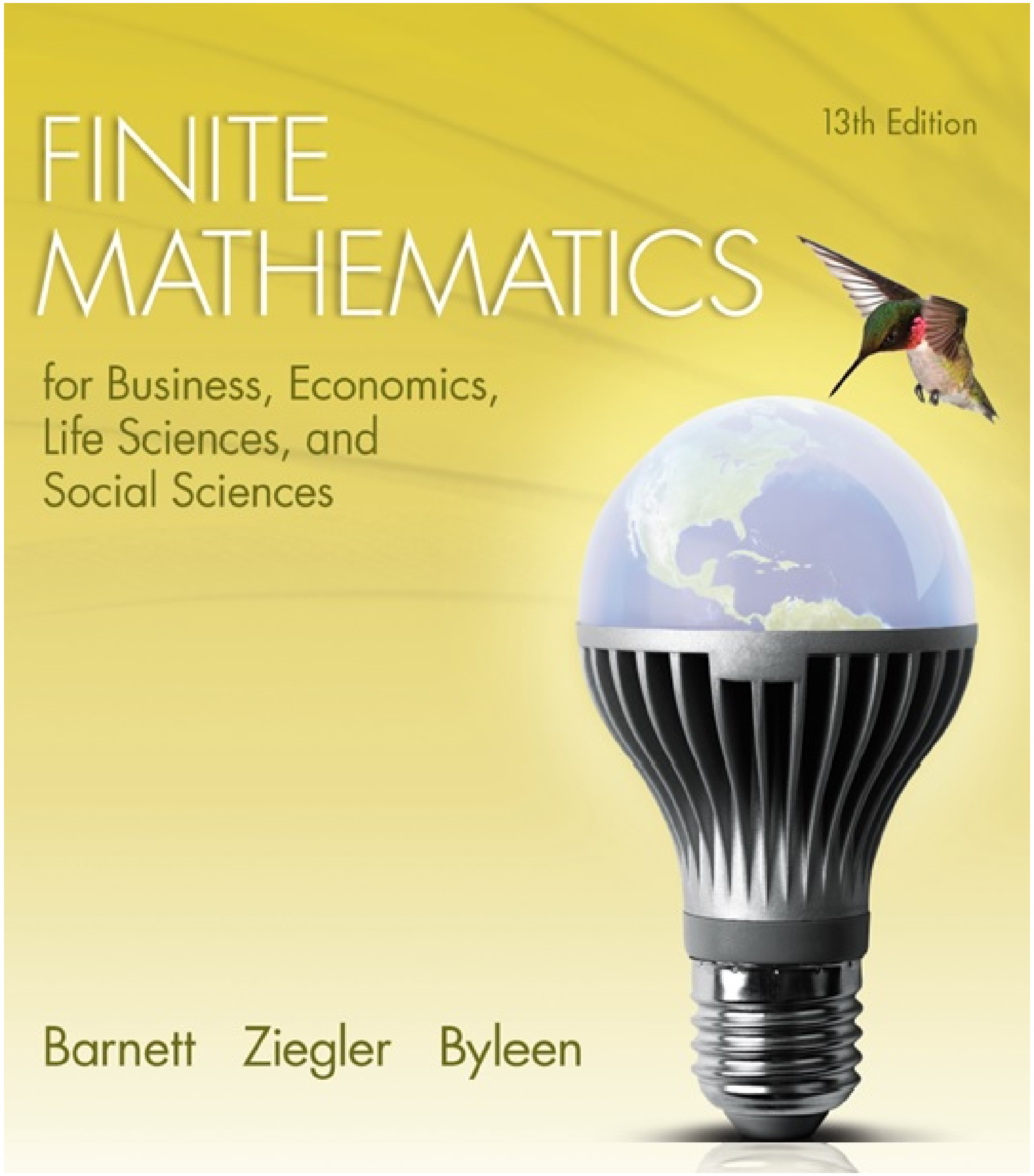


Chapter 8

Probability

Section 3

Conditional Probability, Intersection, and Independence

Learning Objectives for Section 8.3

Conditional Probability, Intersection, and Independence

- The student will be able to calculate conditional probability.
- The student will be able to use the product rule to calculate the probability of the intersection of two events.
- The student will be able to construct probability trees.
- The student will be able to determine if events are independent or dependent.

Conditional Probability, Intersection and Independence

Consider the following problem:

- Find the probability that a randomly chosen person in the U.S. has lung cancer.
- We want $P(C)$. To determine the answer, we must know how many individuals are in the sample space, $n(S)$. Of those, how many have lung cancer, $n(C)$ and find the ratio of $n(C)$ to $n(S)$.

$$P(C) = \frac{n(C)}{n(S)}$$

Conditional Probability

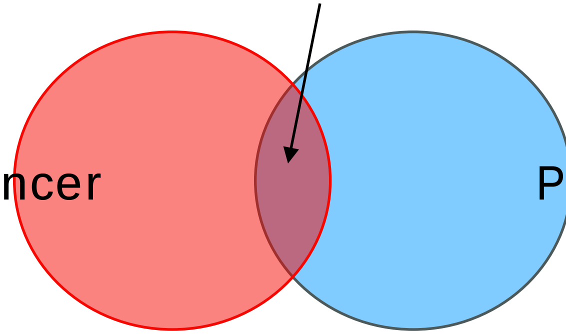
- Now, we will modify the problem: Find the probability that a person has lung cancer, **given** that the person smokes.

  Do we expect the probability of cancer to be the same?
- Probably not, although the cigarette manufacturers may disagree.
- What we now have is called **conditional probability**. It is symbolized by $\boldsymbol{P(C \mid S)}$ and means the probability of lung cancer **assuming or given** that the person smokes.

Conditional Probability Problem

The probability of having lung cancer given that the person smokes is found by determining the number of people who have lung cancer and smoke and dividing that number by the number of smokers.

$$P\left(L \mid S\right) = \frac{n(L \cap S)}{n(S)}$$



Formula for Conditional Probability

Derivation:

$$P\left(L \mid S\right) = \frac{n(L \cap S)}{n(S)}$$

$$P\left(L \mid S\right) = \frac{\dfrac{n(L \cap S)}{n(T)}}{\dfrac{n(S)}{n(T)}}$$

$$P\left(L \mid S\right) = \frac{p(L \cap S)}{p(S)}$$

Dividing numerator and denominator by the total number, $n(T)$, of the sample space allows us to express the conditional probability of L given S as the quotient of the probability of L and S divided by the probability of smoker.

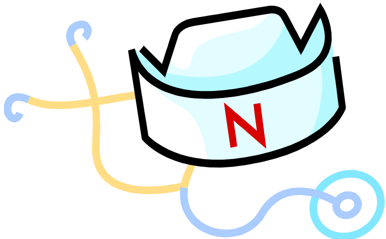
Formula for Conditional Probability

The probability of event A given that event *B* has already occurred is equal to the probability of the intersection of events *A* and *B* divided by the probability of event *B* alone.

$$P\left(A \mid B\right) = \frac{p(A \cap B)}{p(B)} \qquad P(B) \neq 0$$

Example

- There are two majors of a particular college: Nursing and Engineering. The number of students enrolled in each program is given in the table on the next slide. The row total gives the total number of each category and the number in the bottom-right cell gives the total number of students. A single student is selected at random from this college. Assuming that each student is equally likely to be chosen, find :



Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- 1. *P*(Nursing)
- 2. *P*(Grad Student)
- 3. *P*(Nursing and Grad student)
- 4. *P*(Engineering and Grad Student)

Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- 1. P(Nursing) = 100/150 = 2/3
- 2. P(Grad Student) = 60/150 = 2/5
- 3. P(Nursing and Grad student) = 47/150
- 4. P(Engineering and Grad Student) = 13/150

Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an undergraduate student is selected at random, what is the probability that this student is a nurse?

Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an undergraduate student is selected at random, what is the probability that this student is a nurse?
- Restricting our attention to the column representing undergrads, we find that of the 90 undergrad students, 53 are nursing majors. Therefore, $P(N|U)=53/90$

Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an engineering student is selected, find the probability that the student is an undergraduate student.

Example (continued)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an engineering student is selected, find the probability that the student is an undergraduate student.
- Restricting the sample space to the 50 engineering students, 37 of the 50 are undergrads, indicated by the red cell. Therefore, $P(U|E) = 37/50 = 0.74$.

Derivation of General Formulas for $P(A \cap B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \mid B)P(B) = P(A \cap B)$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} \rightarrow P(B \mid A)P(A) = P(B \cap A)$$

$$P(A \cap B) = P(B \cap A) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Example
Solution

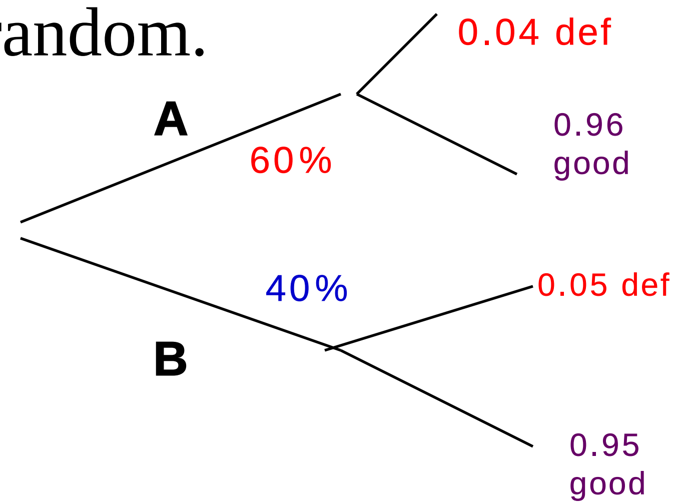
- Two cards are drawn without replacement from an ordinary deck of cards . Find the probability that two clubs are drawn in succession.

Since we assume that the first card drawn is a club, there are 12 remaining clubs and 51 total remaining cards.

$$P(C_1 \cap C_2) = p(C_1) \cdot p(C_2 | C_1) = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$$

Another Example

Two machines are in operation. Machine A produces 60% of the items, whereas machine B produces the remaining 40%. Machine A produces 4% defective items whereas machine B produces 5% defective items. An item is chosen at random.



What is the probability that it is defective?

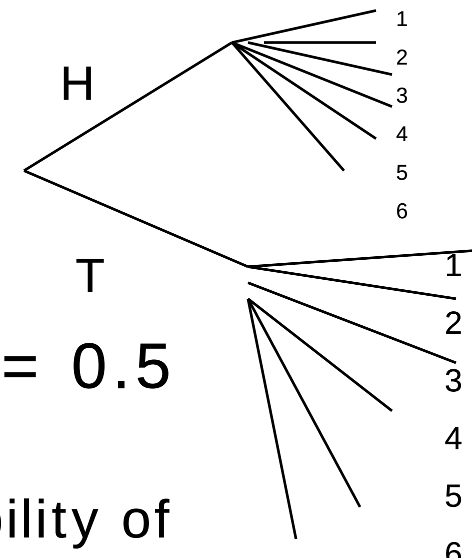
$$P(\text{defective}) = P(A \cap D) + P(B \cap D)$$
$$= P(A)P(D \mid A) + P(B)P(D \mid B)$$
$$= 0.6(0.04) + 0.4(0.05) = 0.044$$

Probability Trees

- In the preceding slide we saw an example of a **probability tree**. The procedure for constructing a probability tree is as follows:
  - Draw a tree diagram corresponding to all combined outcomes of the sequence of experiments.
  - Assign a probability to each tree branch.
  - The probability of the occurrence of a combined outcome that corresponds to a path through the tree is the product of all branch probabilities on the path.

Probability Tree Example Solution

$P(\text{H}) = 0.5$



$P(\text{T}) = 0.5$

Probability of each of these 6 outcomes is 1/6.

A coin is tossed and a die is rolled. Find the probability that the coin comes up heads and the die comes up three.

The outcomes for the coin are{H, T}. The outcomes for the die are {1,2,3,4,5,6}. Using the fundamental principle of counting, we find that there are 2(6)=12 total outcomes of the sample space.

$$P(\text{H and } 3) = (1/2)(1/6) = 1/12$$

Probability Tree Example (continued)

Now, let's look at the same problem in a slightly different way. To find the probability of Heads and then a three on a dice, we have

$$p(H \cap 3) = p(H) \cdot p(3 \mid H)$$

using the rule for conditional probability. However, the probability of getting a three on the die does not depend upon the outcome of the coin toss. We say that these two events are **independent**, since the outcome of either one of them does not affect the outcome of the remaining event.

$$p(H \cap 3) = p(H) \cdot p(3 \mid H) = p(H) \cdot p(3)$$

Independence

Two events are **independent** if

$$p(A \mid B) = p(A)$$

$$p(B \mid A) = p(B)$$

$$p(A \cap B) = p(A)\,p(B)$$

All three of these statements are equivalent.

Otherwise, A and B are said to be **dependent**.

Examples of Independence

1. Two cards are drawn in succession **with replacement** from a standard deck of cards. What is the probability that two kings are drawn?

$$P(K_1 \cap K_2) = p(K_1) \cdot p(K_2)$$

$$= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$$

2. Two marbles are drawn **with replacement** from a bag containing 7 blue and 3 red marbles. What is the probability of getting a blue on the first draw and a red on the second draw?

$$p(B \cap R) = p(B) \cdot p(R)$$

$$= \frac{7}{10} \cdot \frac{3}{10} = \frac{21}{100} = 0.21$$

Dependent Events

- Two events are **dependent** when the outcome of one event affects the outcome of the second event.
- **Example:** Draw two cards in succession **without replacement** from a standard deck. Find the probability of a king on the first draw and a king on the second draw.

Dependent Events

- Two events are **dependent** when the outcome of one event affects the outcome of the second event.
- **Example:** Draw two cards in succession **without replacement** from a standard deck. Find the probability of a king on the first draw and a king on the second draw.
- **Answer:**

$$P(K_1 \cap K_2) = p(K_1) \cdot p(K_2 | K_1)$$

$$= \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

Another Example of Dependent Events

Are smoking and lung disease related?

Step 1. Find the probability of lung disease.

$P(L) = 0.15$ (row total)

Step 2. Find the probability of being a smoker

$P(S) = 0.31$ (column total)

Step 3. Check

$P(L \cap S) = 0.12 \neq P(L) \cdot P(S)$

L and S are **dependent.**

	Smoker	Non-smoker
Has Lung Disease	0.12	0.03
No Lung Disease	0.19	0.66

Summary of Key Concepts

- Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

- A and B are independent if and only if $P(A \cap B) = P(A)\,P(B)$
- If A and B are independent events, then
  $P(A|B) = P(A)$ and $P(B|A) = P(B)$
- If $P(A|B) = P(A)$ or $P(B|A) = P(B)$, then A and B are independent.
- If $E_1, E_2, \ldots, E_n$ are independent, then
  $P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)\,P(E_2) \ldots P(E_n)$