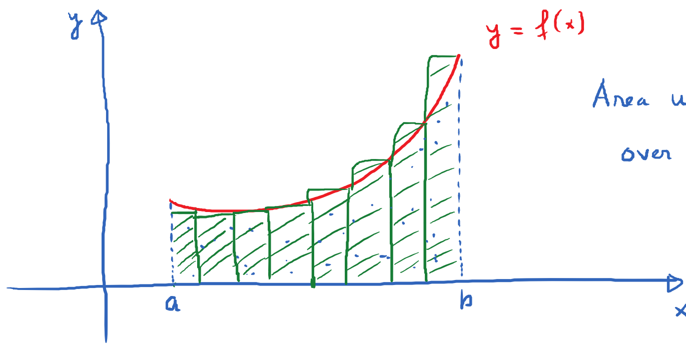
5.1. and 5.2. Areas and the definite integral

Wednesday, August 9, 2017 7:33 AM



Area under graph of $y = f(x)$ over $[a, b]$

Summation Notation (Sigma Notation)

summand

$$1 + 2 + 3 + 4 + 5 + \dots + 1000 = \sum_{i=1}^{1000} (i)$$

index

$$\boxed{\sum_{i=1}^{9999} (i^2 + i)} = 2 + 6 + 12 + \dots (9999^2 + 9999)$$

$$\boxed{\sum_{i=1}^{1000} \frac{1}{i^2}} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{1000000}$$

Summation Formula.

$$\begin{array}{rl} & 1 + 2 + 3 + 4 + \dots + 100 = S \\ + & 100 + 99 + 98 + 97 + \dots + 1 = S \\ \hline & \underbrace{101 + 101 + 101 + 101 + \dots + 101} = 2S \end{array}$$

$$100 \cdot 101 = 2S$$

$$S = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050$$

1)

$$\begin{array}{rl} & 1 + 2 + 3 + \dots + n = S \\ + & n + (n-1) + (n-2) + \dots + 1 = S \\ \hline & \underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1)} = 2S \end{array}$$

$$n \cdot (n+1) = 2S \longrightarrow S = \frac{n(n+1)}{2}.$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$\boxed{\quad i = 1 \qquad\qquad 2 \quad}$$

E.g. $\displaystyle\sum_{i=1}^{20} i = \frac{20\cdot(21)}{2} = 210$

(2) c : constant. $\displaystyle\boxed{\sum_{i=1}^{n} c = n\cdot c}$

$$\underbrace{c + c + c + \cdots + c}_{n}$$

(3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

E.g. $\displaystyle\sum_{i=1}^{10} i^2 = \frac{10\cdot 11\cdot 21}{6} = 385$

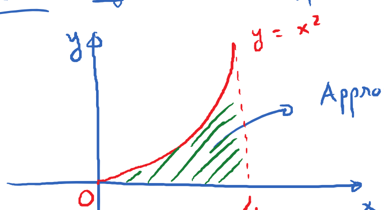
E.g. $\displaystyle\sum_{i=1}^{10} (i+2)^2 = \sum_{i=1}^{10}(i^2+4i+4) = \sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 4i + \sum_{i=1}^{10} 4$

$$= 385 + 4\sum_{i=1}^{10} i + \sum_{i=1}^{10} 4$$

$$= 385 + \frac{4\cdot 10\cdot(11)}{2} + 40$$

$$= 385 + 220 + 40 = 645$$

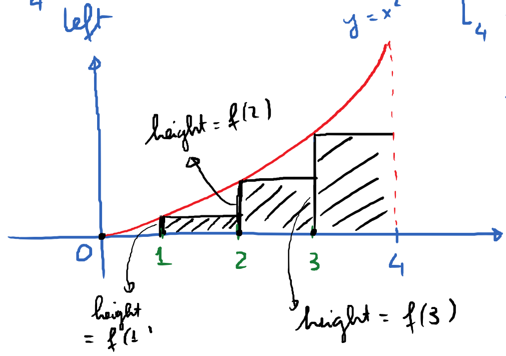
Area E.g. Consider $f(x) = x^2$ over the interval $[0,4]$



Approximate this area using L_4.

and using R_4.

L_4 : a left Riemann sum with 4 subintervals



L_4 = Sum of these rectangles areas.

$= 1\cdot f(1) + 1\cdot f(2) + 1\cdot f(3)$

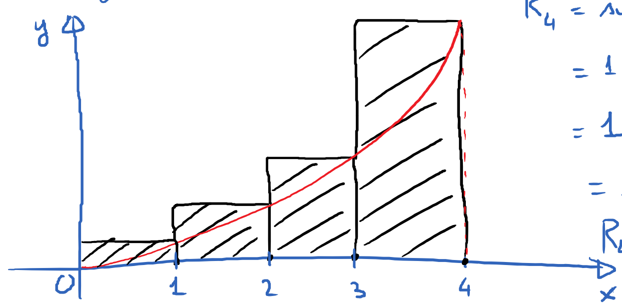
$= 1\cdot 1 + 1\cdot 4 + 1\cdot 9$

$= 14$

$L_4 = 14 \longrightarrow$ underestimate.

$L_4 = 14 <$ Area

R_4: Right Riemann Sum with 4 subintervals



R_4 = sum of areas of rectangles

$= 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4)$

$= 1 \cdot [f(1) + f(2) + f(3) + f(4)]$

$= 1 \cdot [1 + 4 + 9 + 16]$

$R_4 = 30 \longrightarrow$ overestimate

$L_4 = 14 < \text{Area} < R_4 = 30$

$n = 20$ (Divide $[0,4]$ into 20 subintervals)

$$L_{20} = 19.76 < \text{Area} < R_{20} = 22.96$$

$n = 50$

$$L_{50} = 20.6976 < \text{Area} < R_{50} = 21.9776$$

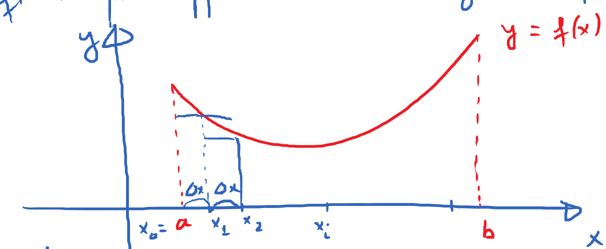
$n = 100$

$$L_{100} = 21.0144 < \text{Area} < R_{100} = 21.6544$$

$$L_n < \text{Area} < R_n$$

$$\lim_{n\to\infty} \boxed{L_n} = \lim_{n\to\infty} \boxed{R_n} = \text{Area}.$$

Left Endpoint approximation and Right Endpoint Approximation.



$f(x)$ over $[a, b]$

left endpoint

(1) Divide the interval $[a, b]$ into n subintervals of equal length.

Each subinterval has length : $\Delta x = \dfrac{b-a}{n}$.

(2) Draw rectangles from the left endpoints of these subintervals.

First left endpoint : $x_0 = a$.

2^{nd} left endpoint : $x_1 = a + \Delta x$

3^{rd} left endpoint : $x_2 = a + 2\Delta x$

$\vdots$

i^{th} left endpoint : $x_i = a + i\Delta x$

(n^{th} left endpoint : $x_{n-1} = a + (n-1)\Delta x$

③ Calculate height corresponding these left endpoints.

$f(x_0)$; $f(x_1)$, $f(x_2)$, ..., $f(x_{n-1})$

④ Calculate the areas of small rectangles.

$f(x_0)\Delta x$, $f(x_1)\Delta x$, $f(x_2)\Delta x$,, ... $f(x_{n-1})\Delta x$.

⑤

L_n = Sum areas of rectangles.

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$\boxed{L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x}$$

Formula for R_n : right endpoint approximation

$$\boxed{R_n = \sum_{i=1}^{n} f(x_i)\Delta x}$$

It turns out that if f is continuous,

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$ = exact area under graph of f on $[a,b]$

Let's apply this for $f(x) = x^2$. on $[0,4]$.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x .$$

$f(x_{i-1}) = x_{i-1}^2$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n} .$$

$$x_{i-1} = a + (i-1)\Delta x = 0 + \boxed{(i-1)\cdot\frac{4}{n}} .$$

$$L_n = \sum_{i=1}^{n} \boxed{f\left(\boxed{(i-1)\cdot\frac{4}{n}}\right)} \cdot \frac{4}{n}$$

$$L_n = \sum^{n} \left((i-1)\cdot\frac{4}{n}\right)^2 \cdot \frac{4}{n} .$$

$$L_n = \sum_{i=1}^{n} \left((i-1)\cdot \frac{4}{n} \right)^2 \cdot \frac{4}{n}.$$

$$L_n = \sum_{i=1}^{n} (i-1)^2 \cdot \frac{16}{n^2} \cdot \frac{4}{n} = \sum_{i=1}^{n} (i-1)^2 \cdot \boxed{\frac{64}{n^3}}$$

doesn't depend on i

$$L_n = \frac{64}{n^3} \sum_{i=1}^{n} (i-1)^2$$

$$L_n = \frac{64}{n^3} \cdot \sum_{i=1}^{n} (i^2 - 2i + 1) = \frac{64}{n^3} \cdot \left[\boxed{\sum_{i=1}^{n} i^2} - 2\boxed{\sum_{i=1}^{n} i} + \sum_{i=1}^{n} 1 \right]$$

$$L_n = \frac{64}{n^3} \cdot \left[\frac{n\cdot(n+1)\cdot(2n+1)}{6} - \cancel{2}\cdot \frac{n(n+1)}{\cancel{2}} + n \right]$$

$$L_n = \frac{64}{n^3} \cdot \left[\frac{n(n+1)(2n+1)}{6} - n(n+1) + n \right]$$

$$L_n = \frac{64n(n+1)(2n+1)}{6n^3} - \frac{64n(n+1)}{n^3} + \frac{64n}{n^3}.$$

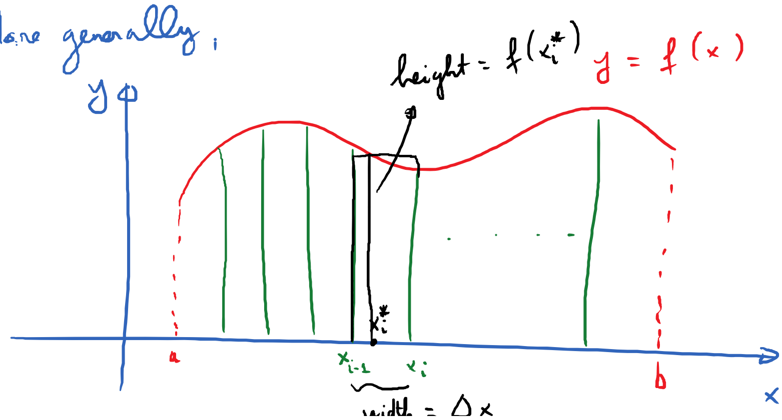
$$L_n = \boxed{\frac{32(n+1)(2n+1)}{3n^2}} - \underbrace{\cancel{\frac{64(n+1)}{n^2}}}_{n\to\infty} \xrightarrow{} 0 + \underbrace{\cancel{\frac{64}{n^2}}}_{n\to\infty} \xrightarrow{} 0.$$

$$\text{Exact Area} = \lim_{n\to\infty} L_n = \boxed{\frac{64}{3}}$$

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad [0,4]$$

$$\frac{64}{3} - 0 = \frac{64}{3} - 0 = \frac{64}{3}$$

More generally,



width $= \Delta x$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\underbrace{f(x_i^*)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}}_{\text{area of } i^{th} \text{ rectangle}} = \text{exact area}$$

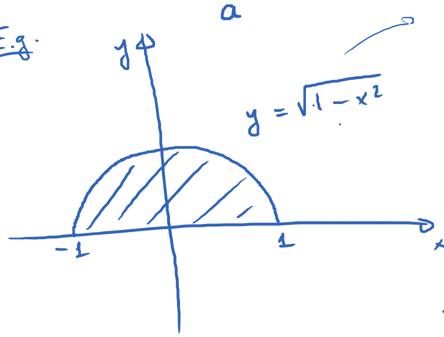
This limit is defined to be the definite integral of $f(x)$ over $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x = \text{exact area under } f(x) \text{ over } [a,b]$$

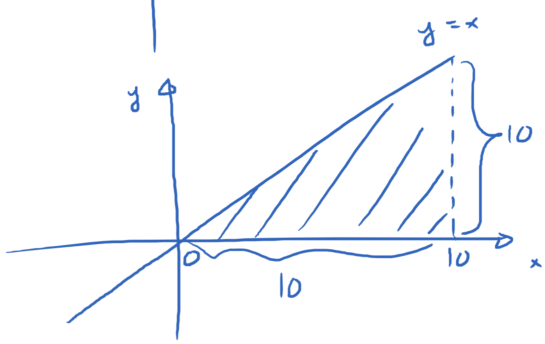
E.g.

$y^2 = 1-x^2$

$x^2+y^2=1$

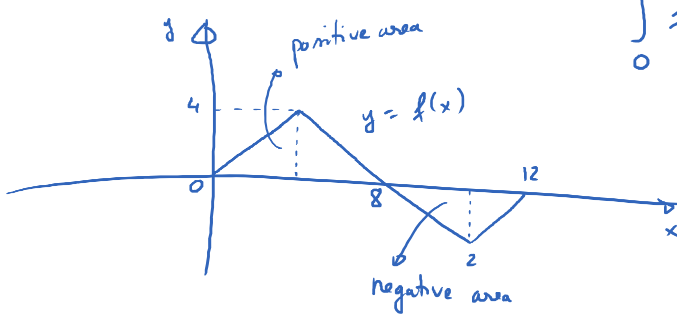


$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$



$$\int_0^{10} x\,dx = 50$$

$$\int_0^{12} f(x)\,dx = 16-4 = 12.$$



Some properties of the definite integrals:

(1) $\displaystyle\int_a^a f(x)\,dx = 0$

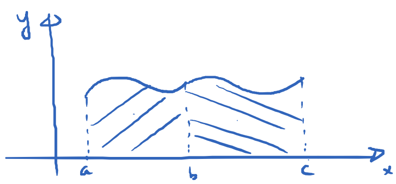
(2) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

(3) $\displaystyle\int^b k\cdot f(x)\,dx = k\int^b f(x)\,dx$

③ $$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

④ $$\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

⑤ $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Wednesday, August 9, 2017 7:56 AM