2.4. Continuity

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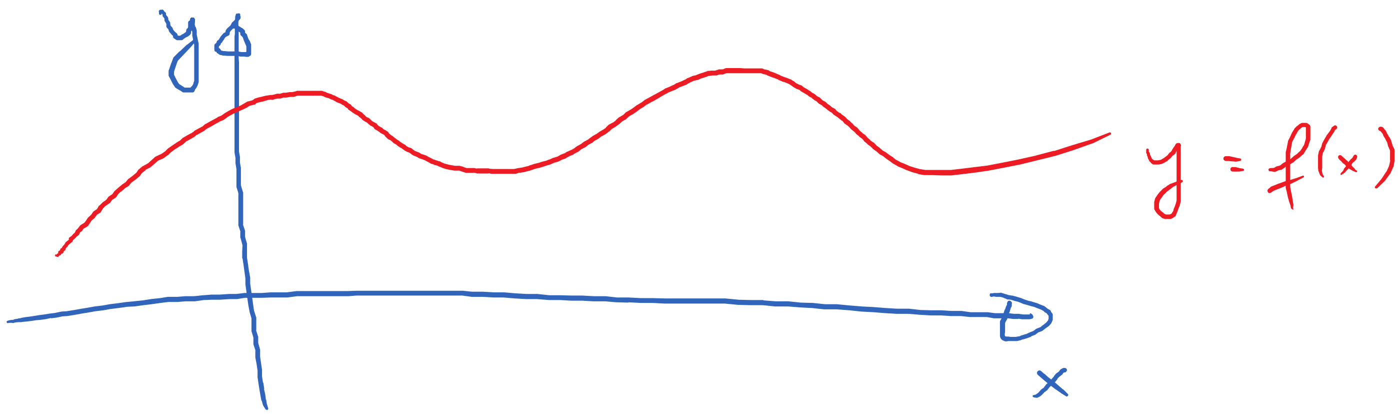
Goals: 1 Understand Continuity using limits

2 Classify different types of discontinuity.



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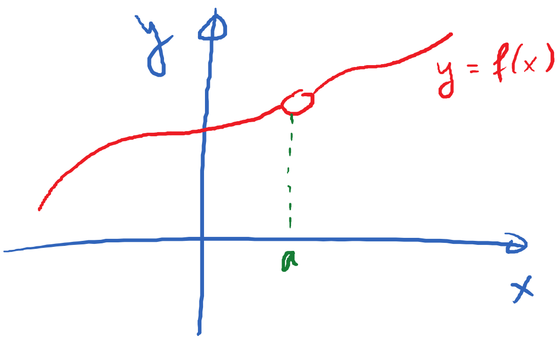
What does it mean for $y = f(x)$ to be continuous at a point $x = a$?

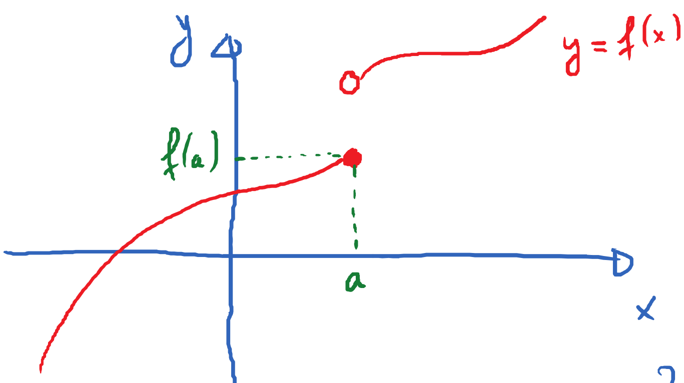
When does f fail to be continuous at $x = a$?



If $f(a)$ is undefined, then f will not be continuous at a.

$1^{\underline{st}}$ requirement for continuity:

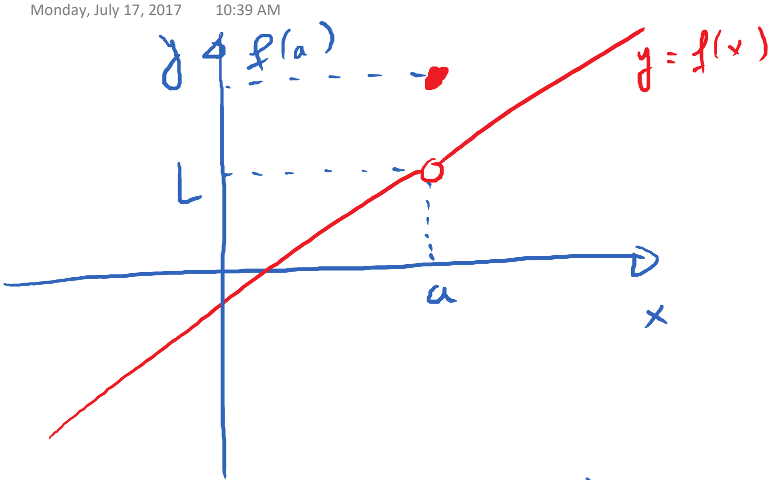
$\boxed{f(a) \text{ is defined}}$



If $\lim_{x \to a} f(x)$ DNE, then f will not be continuous at a.

$2^{\underline{nd}}$ requirement for continuity at $x = a$ is

$\boxed{\lim_{x \to a} f(x) \text{ exist}}$



3rd requirement for continuity

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Def: A function $f(x)$ is continuous at a point $x = a$ if and only if the following requirements are satisfied

1. $f(a)$ must be defined
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$

E.g. Let $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$

Use the above definition to demonstrate that f is continuous at $x = 0$.

1. Is $f(0)$ defined? Yes, $f(0) = -1$ ✓

2. Does $\lim_{x \to 0} f(x)$ exist? $\left\{ \begin{array}{l} \lim_{x \to 0^-} f(x) = -1 \\ \lim_{x \to 0^+} f(x) = -1 \end{array} \right\}$

   Yes, $\lim_{x \to 0} f(x)$ exist and $= -1$

3. $\lim_{x \to 0} f(x) = f(0)$ ✓

f is continuous at $x = 0$

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Ex. ① Is $g(x) = \frac{2x^2 - 5x + 3}{x - 1}$ continuous at $x = 1$?

No. B/c g is not defined at 1.

② $g(x) = \frac{x^2 - 1}{x - 1}$ continuous at $x = 1$?

No. B/c g is not defined at 1

③ $k(x) = \begin{cases} 3x & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$ Is k continuous at $x = 1$?

No. B/c $\lim_{x \to 1} k(x)$ DNE

3 $h(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Is this function cont. at $x = 0$?

Yes. $\left.\begin{array}{l} \displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1 \\ f(0) = 1. \end{array}\right\} \displaystyle\lim_{x \to 0} \frac{\sin x}{x} = f(0)$ ✓

Fact: Any polynomial function is continuous at every real number.

E.g. $g(x) = x^5 - 4x^4 + 3x^3 - 2x^2 + x + 1$

g is continuous everywhere

Fact: Rational functions are continuous at every point in their domains

E.g. $f(x) = \dfrac{1}{x-5}$; Domain $= (-\infty, 5) \cup (5, \infty)$

f is continuous on $(-\infty, 5) \cup (5, \infty)$

Fact: Radical functions are continuous at every point in their domains.

$f(x) = \sqrt{x-5}$ Domain: $[5, \infty)$

$x - 5 \geq 0$

$x \geq 5$

$g(x) = \sqrt[3]{x}$ Domain: $(-\infty, \infty)$

E.x.

1 Find the interval of continuity for

$$f(x) = \frac{x-7}{x^2+10}.$$

2 $g(x) = \sqrt{x^3 - x}$

1 Denominator: $x^2+10 = 0$. It has no real solution.

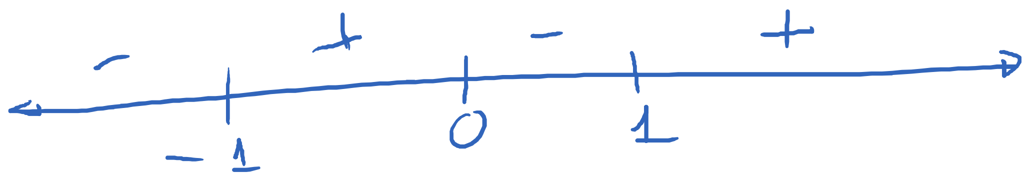
Domain: $(-\infty,\infty) \leftarrow$ interval of continuity.

② $g(x) = \sqrt{x^3 - x}$. $[-1, 0] \cup [1, \infty)$ → interval of continuity

To find domain : Solve $x^3 - x \geq 0$

$$x(x^2 - 1) \geq 0$$

$$x(x+1)(x-1) \geq 0$$



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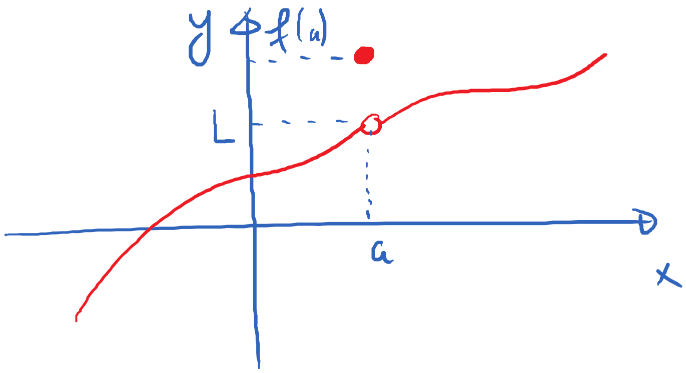
Types of Discontinuity.

① Removable Discontinuity



f has a removable discontinuity at $x = a$ if

① $\lim_{x \to a} f(x)$ exists

But ② $\lim_{x \to a} f(x) \neq f(a)$

$$f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2} & \text{if } x \neq -2 \\ -1 & \text{if } x = -2 \end{cases}$$

this will make it continuous.

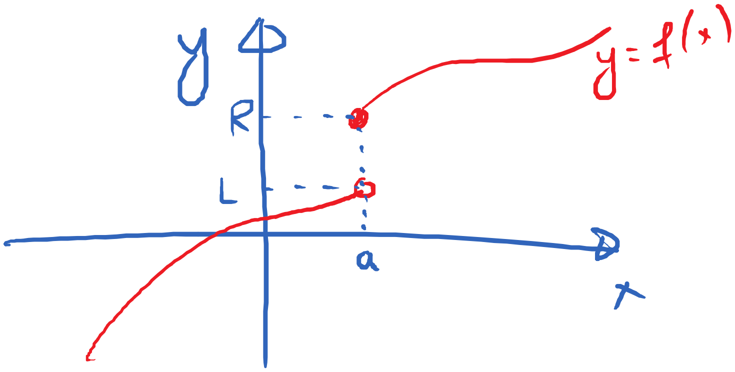
Show this function has a removable discontinuity at $x = -2$.

different

$f(-2) = 1$.

$$\lim_{x \to -2} \frac{x^2+3x+2}{x+2} \stackrel{\frac{0}{0}}{=} \lim_{x \to -2} \frac{\cancel{(x+2)}(x+1)}{\cancel{x+2}} = \lim_{x \to -2} (x+1) = -1$$

② Jump Discontinuity.



We say that f has a jump discontinuity if

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

E.g. $f(x) = \begin{cases} x\sin(x) & \text{if } x < \pi \\ x\cos(x) & \text{if } x \geq \pi \end{cases}$

Show that f has a jump discontinuity at $x = \pi$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi} (x \sin(x)) = \pi \cdot \sin(\pi) = 0$$

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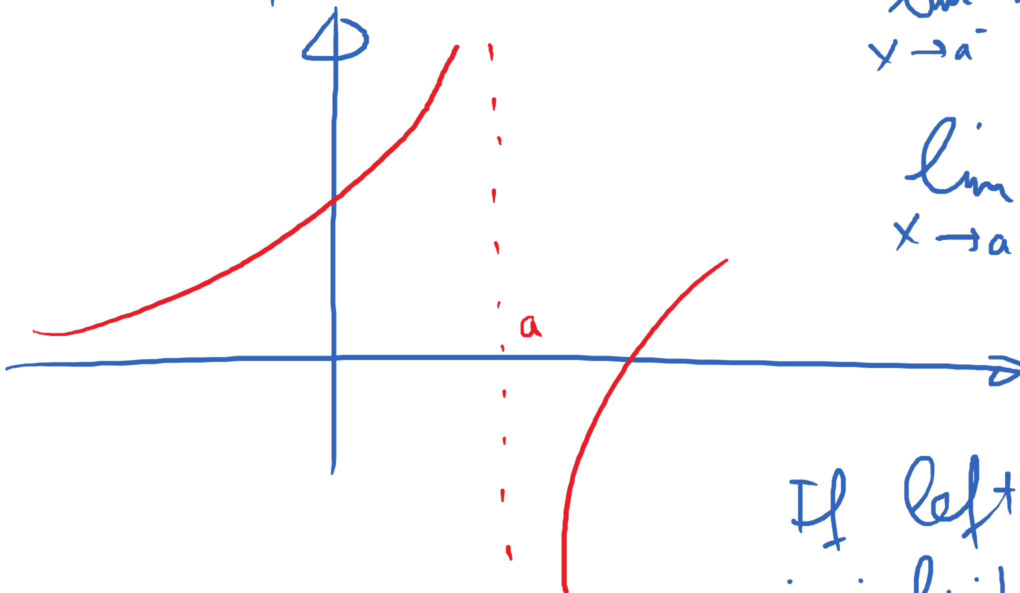
$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi} (x \cdot \cos(x)) = \pi \cdot \cos(\pi) = -\pi .$$

3 Infinite Discontinuity.

$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$



If left or right limit is infinite, then f has an infinite discontinuity at that point