Derivatives of Exponential and Logarithmic Functions

Goals: (1) Derivatives of Exponential Functions

(2) Derivatives of Logarithmic Functions

(3) Logarithmic Differentiation.

Review of basic differentiation formulas

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$$

u is a function of x

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sqrt{u}\right] = \frac{1}{2\sqrt{u}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\frac{1}{u}\right] = -\frac{1}{u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin u\right] = \cos u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos u\right] = -\sin u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan u\right] = \sec^2 u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sec u\right] = \sec u \tan u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

(1) Derivatives of Exponential Function.

① Derivatives of Exponential Function.

Base e:

$e \approx 2.71828$

$f(x) = e^x$

$f(0) = e^0 = 1.$

$f(1) = e$; $f(2) = e^2$

$f(-1) = \frac{1}{e}$; $f(-2) = \frac{1}{e^2}$.

If $f(x) = e^x$, then $f'(x) = e^x$.

In Leibnitz notation: $\frac{d}{dx}[e^x] = e^x$

If u is a function of x; $\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$.

In Newton's notation: $\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x)$.

E.g. Find the given derivative.

(a) $\frac{d}{dx}[e^{\pi}] = 0$

(b) $\frac{d}{dx}[e^{-x}] = e^{-x} \cdot (-1) = \boxed{-e^{-x}}$

(c) $\frac{d}{dx}\left[e^{\underbrace{\boxed{\sec x}}_{u}}\right] = e^{\sec x} \cdot \sec x \cdot \tan x$

(d) $\frac{d}{dx}\left[x \cdot \boxed{e^{2x}}\right] = 1 \cdot e^{2x} + x \cdot 2e^{2x}$

$\left(e^{\overset{u}{\boxed{2x}}}\right)' = e^{2x} \cdot 2$

Exponential Function with base other than e.

$f(x) = 2^x$; $f'(x) = ?$

Given $a > 0$; $a \neq 1$; $a \neq e$.

$f(x) = a^x$.

Formula: $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$

u: function of x.

$\frac{d}{dx}[a^u] = a^u \cdot \ln(a) \cdot \frac{du}{dx}$

u: function of x.

$$\frac{d}{dx}\left[a^u\right] = a^u \cdot \ln(a) \cdot \frac{du}{dx}$$

E.g. $\frac{d}{dx}\left[2^x\right] = 2^x \cdot \boxed{\ln(2)}$

Why? $\frac{d}{dx}\left[\boxed{2^x}\right]$

$$y = 2^x = \left(\boxed{e^{\ln 2}}\right)^x = \boxed{e^{x\ln 2}}$$

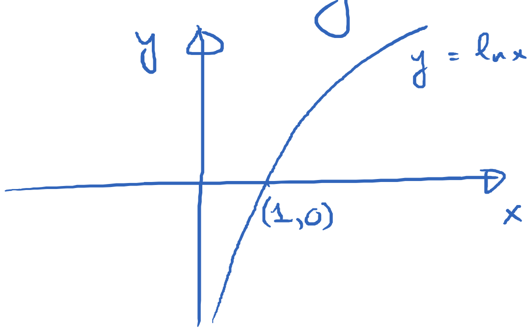
$$(2^x)' = \left(e^{\underbrace{\boxed{x\ln 2}}_{u}}\right)' = e^{x\ln 2} \cdot \ln 2 = 2^x \cdot \ln 2.$$

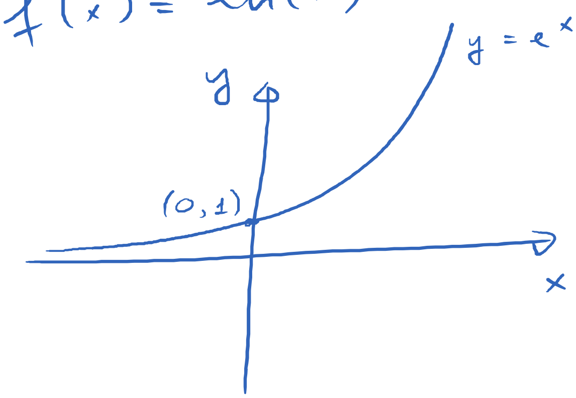
Ex. (a) $\frac{d}{dx}\left[5^{\cos(3x)}\right]$ (b) $\frac{d}{dx}\left[x^{\pi} \cdot \pi^{x}\right]$

Solved in class

(2) Derivatives of Log Functions.

Natural log function: $f(x) = \ln(x)$





$\ln(1) = 0$. $\ln(e) = 1$. $\ln(e^2) = 2$

$$\ln\left(e^{\text{Stuff}}\right) = \text{Stuff}$$

$$e^{\ln(\text{Stuff})} = \text{Stuff}.$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad ; \quad \frac{d}{dx}\left(\ln|u|\right) = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}.$$

u : function of x

Why: $\boxed{y = \ln x}$; $\frac{dy}{dx} = ?$

$\longleftrightarrow \boxed{e^y = x} \longrightarrow$ Implicit Differentiation:

$$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$$

$$e^y \cdot \frac{dy}{dx} = 1 \longleftrightarrow \boxed{\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}}$$

E.g. $\frac{d}{dx}\left[\ln\left(\sin x\right)\right] = \frac{\cos x}{\sin x} = \cot x$

$$\frac{d}{dx}\left[\ln\left(\cos x\right)\right] = \boxed{\frac{-\sin x}{\cos x}} = -\tan x.$$

Useful Properties of the ln function:

$$\boxed{\begin{aligned} \ln(u\cdot v) &= \ln(u) + \ln(v) \\ \ln\left(\frac{u}{v}\right) &= \ln(u) - \ln(v) \\ \ln\left(u^a\right) &= a\cdot\ln(u) \end{aligned}}$$

$e^u \cdot e^v = e^{u+v}$

$$\begin{aligned} \ln(u^{2}) &= \ln(u\cdot u) \\ &= \ln(u) + \ln(u) \\ &= 2\cdot\ln u \end{aligned}$$

E.g. $\frac{d}{dx}\left[\ln\left(\underbrace{\boxed{\frac{2x+3}{4x+5}}}_{}\right)\right]$ (u)

$$= \frac{d}{dx}\left[\boxed{\ln(2x+3)} - \ln(4x+5)\right]$$

$$= \frac{2}{2x+3} - \frac{4}{4x+5}$$

E.g. $\frac{d}{dx}\left(\ln\left[(3x+2)^{2017}\right]\right) = \frac{d}{dx}\left[2017 \cdot \ln(3x+2)\right]$

$$= 2017 \cdot \frac{d}{dx}\left(\ln(3x+2)\right)$$

$$= 2017 \cdot \frac{3}{3x+2} = \frac{6051}{3x+2}.$$

Ex. Find the derivative (Remember to use the properties of ln)

ⓐ $\ln\left(\sqrt{5x+7}\right)$

ⓑ $\ln\left(\frac{(2x+1)^5}{\sqrt{x^2+1}}\right)$

Solved in class.

Base $\neq e$: $f(x) = \log_2(x)$

$b \neq 0;\ b > 1;\ b \neq e.$

$$\frac{d}{dx}\left[\log_b x\right] = \frac{1}{x \cdot \ln(b)}.$$

$$\frac{d}{dx}\left[\log_b u\right] = \frac{1}{u \cdot \ln(b)} \cdot \frac{du}{dx}.$$

E.g. $\frac{d}{dx}\left[\log_2(x)\right] = \frac{1}{x\ln(2)}$

E.g. $\frac{d}{dx}[\log_2 \ldots] \quad \ldots x\ln(2)$

$$\frac{d}{dx}\left[\log_{10}(x^3+1)\right] = \frac{1}{(x^3+1)\ln(10)} \cdot 3x^2 = \frac{3x^2}{(x^3+1)\ln(10)}$$

Problem: $y = x^x$. $\frac{dy}{dx} = ?$

Logarithmic Differentiation:

① Take the ln of both sides:

$$\ln y = \ln x^{\boxed{x}}$$

$$\ln y = x \cdot \ln x$$

② Implicit Differentiation: take the derivative w.r.t. x of both sides

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \cdot \ln x]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + \frac{1}{x} \cdot x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + 1$$

③ Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

④ Replace y by x^x: So,

$$\boxed{\frac{dy}{dx} = x^x \cdot (\ln x + 1)}$$

$$\boxed{\frac{dy}{dx} = x^{x} \cdot (\ln x + 1)}$$

Ex: $y = (x^3 + 1)^{\frac{1}{x}}$. $\frac{dy}{dx} = ?$

Solved in class