3.3. Differentiation Rules

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Goals: (1) Power Rule, Sum and Difference Rules

(2) Product and Quotient Rule

(3) Higher order derivatives.

$$\underbrace{f'(x)}_{\frac{dy}{dx}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

E.g. $f(x) = x^2$. $f'(x) = 2x$.

What if $f(x) = x^3$; $f(x) = x^4$; in general, $f(x) = x^n$?

Power Rule.

$$f(x) = x^3.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \quad \left(\frac{0}{0}\right)$$

Pascal triangle.

1
1 → 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(x+h)^2 = x^{\boxed{2}} + 2x^1h^1 + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3 \checkmark$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

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$$\lim_{h \to 0} \frac{\cancel{x^3} + \boxed{3x^2 h + 3x h^2 + h^3} \cancel{- x^3}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2$$

$f(x) = x^3$. $f'(x) = 3x^2$.

Leibnitz notation: $\frac{d}{dx}[x^3] = 3x^2$

$\frac{d}{dx}[x] = 1$; $\frac{d}{dx}[x^2] = 2x$; $\frac{d}{dx}[x^3] = 3x^2$

$\frac{d}{dx}[x^4] = 4x^3$

In general, $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$ ⟵ Power Rule.

If n is any real # ; $\frac{d}{dx}[x^n] = n x^{n-1}$.

$\boxed{\frac{d}{dx}[x^{\frac{1}{2}}] = \frac{1}{2} \cdot x^{-\frac{1}{2}}}$. $\frac{d}{dx}[x^{\pi}] = \pi x^{\pi - 1}$

$$\frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}\cdot x^{-\frac{1}{2}} = \frac{1}{2\,x^{\frac{1}{2}}}$$

$$\boxed{\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}}$$

Ex. Find the given derivatives:

ⓐ $\frac{d}{dx}\left[x^{2017}\right]$ ⓑ $\frac{d}{dx}\left[x^{e}\right]$ ⓒ $\frac{d}{dx}\left[e^{2017}\right]$

ⓓ $\frac{d}{dx}\left[x^{-6}\right]$ ⓔ $\frac{d}{dx}\left[\frac{1}{x}\right]$ ⓕ $\frac{d}{dx}\left[\sqrt[3]{x^2}\right]$

ⓖ $\frac{d}{dx}\left[\frac{2}{\sqrt[3]{x^2}}\right]$

Solved in class.

Constant Multiple Rule. Sum / difference Rule

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

In prime notation: $\left(c\,f(x)\right)' = c \cdot f'(x)$

E.g. $\frac{d}{dx}\left[\pi \cdot \sqrt{x}\right] = \pi \cdot \frac{d}{dx}\left[\sqrt{x}\right] = \pi \cdot \frac{1}{2\sqrt{x}}$

$$= \frac{\pi}{2\sqrt{x}}$$

Sum / difference Rule

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

$$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$$

E.g. $\frac{d}{dx}\left[x^3 + \frac{1}{x}\right] = 3x^2 + \left(-\frac{1}{x^2}\right)$
$= 3x^2 - \frac{1}{x^2}$.

E.g. $f(x) = x^3 - 4x^2 + 3x + 6$

(a) Find the equation of the tangent line to the graph of $y = f(x)$ at $(0,6)$.

(b) Find all the x-values for which the graph has a horizontal tangent line.

(a) Slope of tangent line at $(0,6)$ $= f'(0)$

$f'(x) = 3x^2 - 8x + 3$.

$f'(0) = 3 \leftarrow$ Slope.

$y - 6 = 3\cdot(x - 0) \Longrightarrow y - 6 = 3x$

$\boxed{y = 3x + 6}$

(b) Horizontal tangent line $\Rightarrow$ Slope $= 0$

$$\Rightarrow f'(x) = 0$$

$$\boxed{3x^2 - 8x + 3} = 0$$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{8 \pm \sqrt{64 - 36}}{6} = \frac{8 \pm \sqrt{28}}{6}$$

Product Rule

WRONG Product Rule.

$$\cancel{\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right]\cdot\frac{d}{dx}\left[g(x)\right]}$$

$$\cancel{\frac{d}{dx}\left[x^2\cdot x^3\right] = \frac{d}{dx}\left[x^2\right]\cdot\frac{d}{dx}\left[x^3\right] = 2x\cdot 3x^2 = 6x^3}$$

$$\cancel{\frac{d}{dx}\left[x^5\right] = 5x^4}$$

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Correct Version:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \left(\frac{d}{dx}\left[f(x)\right]\right)\cdot g(x) + \left(\frac{d}{dx}\left[g(x)\right]\right)\cdot f(x)$$

In prime notation:

$$\left(f(x)\cdot g(x)\right)' = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

$$(f\cdot g)' = f'\cdot g + g'\cdot f$$

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E.g. $j(x) = \underbrace{(2x^5 - 1)}_{f(x)} \cdot \underbrace{(x^2 + x)}_{g(x)}$

$j'(x) = ?$

$j'(x) = \underbrace{10x^4}_{f'} \cdot \underbrace{(x^2 + x)}_{g} + \underbrace{(2x + 1)}_{g'} \cdot \underbrace{(2x^5 - 1)}_{f}$

Quotient Rule

Wrong version: $\cancel{\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]}}$

Correct Version: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2}$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}$$

$$\text{Quotient} = \frac{\text{low}\cdot d(\text{high}) - (\text{high})\cdot d(\text{low})}{(\text{low})^2}$$

E.g. $h(x) = \dfrac{\overset{f}{3x+1}}{\underset{g}{4x-3}}$. $\quad h'(x) = ?$

$$h'(x) = \frac{\overbrace{(4x-3)}^{g}\cdot \overset{f'}{3} - \overbrace{(3x+1)}^{f}\cdot \overset{g'}{4}}{\underbrace{(4x-3)^2}_{g^2}}$$

$$= \frac{\cancel{12x} - 9 - \cancel{12x} - 4}{(4x-3)^2} = \boxed{\frac{-13}{(4x-3)^2}}$$

Ex. let $g(x) = \dfrac{x^2+4}{x^2-4}$.

Find $g'(x)$. (and simplify)

Higher order derivative:

$f(x) = x^3 + 2x^2$.

$f'(x) = 3x^2 + 4x \longleftarrow$ the first derivative.

The second derivative = the derivative of the first derivative.

$f''(x) = 6x + 4$

The third derivative = the derivative of the second derivative

$f'''(x) = 6$

$f^{(4)}(x) = 0$; $f^{(5)}(x) = 0$

In Leibnitz notation: $\dfrac{dy}{dx} \longrightarrow 1^{st}$ derivative

$\dfrac{d^{\boxed{2}}y}{dx^{\boxed{2}}} \longrightarrow 2^{nd}$ derivative

$\dfrac{d^3y}{dx^3} \longleftarrow 3^{rd}$ derivative.

$\frac{\quad}{dx^3}$ ——— 3 – derivative